A Two-dimensional Numerical Model of Estuarine Circulation: The Effects of Altering Depth and River Discharge

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Steady-state numerical solutions are obtained for a two-dimensional, vertically stratified model of a partially mixed estuary. The boundary at the seaward end of the estuary is considered to be open, with the profiles of salinity, vorticity and streamfunction obtained by extrapolating interior dynamics out to the boundary. A salinity source is maintained at the bottom at the mouth. Zero salt flux is required at a free-slip top and no-slip bottom boundary. Zero salinity and a parabolic velocity profile are maintained at the head of the estuary.

A number of cases are run for various estuarine parameters; the river transport and Rayleigh number being the two parameters that have the most pronounced effect. The river transport is varied by adjusting the mean freshwater velocity, $U_r$. Decreasing $U_r$ allows salt as well as the stagnation or null point to penetrate upstream. The estuarine circulation weakens, but expands over a larger portion of the estuary. The position of the stagnation point, with respect to the seaward boundary, varies as $U_r^{-2/3}$ for $U_r > 1$ cm/s and as $U_r^{-1/3}$ for $U_r < 1$ cm/s. Increasing the Rayleigh number, by deepening the estuarine channel, $H$, results in an increased circulation as well as strong intrusion of salinity and inward migration of the stagnation point. The horizontal location of the stagnation point is found to be proportional to $Ra$ and therefore, varies as $H^3$.

Introduction

Estuaries are the part of the ocean most subject to modification, sometimes to enhance their commercial utility, sometimes as an unanticipated side effect of engineering works elsewhere in their watershed. One of the earliest, and still most common, modifications stems from the attractiveness of estuaries as sites for coastal cities and seaports. Most of the major seaports of the world consist of estuaries improved for large vessel navigation by dredging of entrance bars and channel deepening; however, undesired side effects have sometimes accompanied these improvements. Freshwater supplies on the Delaware River estuary are periodically threatened. Marine borers have invaded previously inhospitable regions in San Francisco Bay. Both of these have resulted from alteration of the salinity distribution as a consequence of channel deepening. The growth of urban areas on estuaries sometimes has more subtle effects. Filling bay marshes for development results in a reduction of the tidal prism, thus inhibiting natural water exchange and flushing processes. River flow modifications for agriculture, flood control or hydroelectric power generation have also had unpredicted
adverse downstream consequences due to alteration of the salinity distribution and circulation in the river estuary. The classic example of this type of misadventure was the diversion of Santee River water into the Cooper River in South Carolina. Readjustment of the estuarine circulation to the new dynamic regime resulted almost immediately in severe shoaling problems in Charleston Harbor. More subtle effects, such as the reduction of water exchange within estuarine embayments because of the regulation of seasonal peaks of river discharge, probably occur, but we know of no clear documentation of this effect.

Mathematical models have come into frequent use in estuaries, especially in application to problems of distribution of heat, salinity and solutes. Nearly all of these models are of the barotropic or vertically integrated numerical type. These are very useful for wave phenomena such as tides and tidal currents, or transports of solutes for which effective dispersion coefficients can be determined empirically. They are not well suited for predicting responses of circulation and salinity distributions to engineering modifications, because estuarine dynamics depend strongly upon the vertical structure of the circulation and stratification and their interaction. Baroclinic models in which vertical variations are retained have been presented by Rattray & Hansen (1962) and Hansen & Rattray (1965) by the method of similarity solutions. Although instructive as to the interpretation and generalization of physical processes observed in estuaries, these models are of limited utility in predicting responses to alterations. A characteristic feature of estuarine flows, the transition from estuarine to riverine dynamics is precluded by the mathematical constraints of the similarity approach.

In this paper we present some results from a two-dimensional, numerical model of the gravitational circulation within estuaries. The model is used to investigate effects of channel deepening and variation of stationary river discharge volumes on the circulation and salinity distribution in estuaries. Turbulent transports of salt and momentum are expressed by Fickian type diffusion coefficients. The vertical structure of circulation and stratification, and their interaction, are retained, in contrast to the model of Harleman et al. (1974) which has inconsistent modeling of density advection. A time-dependent version of the problem has been modeled by Hamilton (1975), but he seems not to have run the model long enough to ascertain that the circulation had come to equilibrium with the density field.

**Governing equations**

The problem considered is that of a steady state, two-dimensional, laterally homogeneous estuary (Pritchard, 1956). The co-ordinate system is Cartesian in $x$ and $z$, where $z$ is positive upward and $x$ increases toward the river. A linear equation of state, $\rho = \rho_0(1 + \beta S)$, is assumed and the Boussinesq approximation (Spiegel & Veronis, 1960) is employed.

The horizontal and vertical momentum balances, continuity of flow and conservation of salinity are:

\begin{align}
    u_t + uu_x + wu_z &= -\rho_0^{-1}P_x + (A_h u_x)_x + (A_s u_z)_z, \\
    w_t + uw_x + w^2_z &= -\rho_0^{-1}P_z + (A_h w_x)_x + (A_s w_z)_z - \beta g S,
\end{align}

\begin{align}
    u_x + w_z &= 0, \\
    S_t + u S_x + w S_z &= (K_h S_x)_x + (K_s S_z)_z.
\end{align}

where $u$ and $w$ are the horizontal and vertical components of velocity respectively, $P$ is the hydrostatically reduced pressure, $S$ is the salinity field, $\beta$ is the coefficient of 'salt contraction', $\rho_0$ is the density of fresh water, $A_h$, $A_s$, and $K_h$, $K_s$ are the horizontal and vertical exchange coefficients of momentum and salt, respectively, and $g$ is the gravitational acceleration.
Tidal fluctuations have been averaged out; however, the tides are considered to be the primary source of energy for turbulent mixing. The exchange coefficients therefore represent a measure of the strength of tidal mixing. For simplicity, these coefficients are chosen to be constant.

The vorticity and salt equations corresponding to (1) are:

\[ \eta_t = -\nabla \cdot (\psi \eta) + A \eta_{xx} - \beta g S_x, \]  
\[ S_t = -\nabla \cdot (\psi S) + K_S \eta_{xx} + K_S S_{xx}, \]  
where \( \eta \) is a streamfunction with \( \partial \epsilon \) being a unit vector in the +r direction, \( \eta = A \nabla \psi \) is the vorticity, \( \nabla \cdot \psi \) is the Laplacian, and \( \partial^2 \omega^2 \) is the two-dimensional Laplacian operator.

Non-dimensional equations, corresponding to equation (2), are obtained by scaling \( t \) and \( \eta \) by \( \tau_d = H^2/K \), and \( \tau_d^{-1} \), respectively, \( x \) and \( z \) by \( H \), \( \psi \) by \( K \), and \( S \) by \( \Delta S_h \). \( \tau_d \) is the vertical diffusive time scale, \( H \) is the depth of the estuary, and \( \Delta S_h \) is the horizontal salinity difference between the river and mouth of the estuary. In non-dimensional form the vorticity and salt equations are:

\[ \eta_t = -\nabla \cdot (\psi \eta) + (A \eta_{xx} + \beta S_x), \]  
\[ S_t = -\nabla \cdot (\psi S) + K \eta_{xx} + K \eta_{xx}, \]  
where \( \eta = \partial^2 \psi = \psi_{xx} + \psi_{zz}, \) \( Ra = B g \Delta S_h H^2/(A \nabla) \) is the estuarine Rayleigh number, \( \sigma = A \nabla \) is the Prandtl number, \( A = A_h/A \), and \( K = Ka/K_h \). Non-dimensionalizing both horizontal and vertical distances by the estuarine depth, \( H \), while arbitrary, forces an aspect ratio, \( \epsilon = H/L \), to enter only through the boundary conditions. Here, \( L \) is defined as the computational length of the model estuary, that is, the location of the upstream boundary. This length should not be confused with the dynamical length of the estuary, \( L_d \), roughly equivalent to the extent of salinity intrusion. The determination of the dynamical length is a major object of analysis.

Boundary conditions

The boundary conditions to be satisfied at the river end are zero salinity and a parabolic velocity profile (consistent with constant density and viscosity) having a transport per unit width \( T_r = U_r H \), where \( U_r \) is the vertically averaged river flow per unit width. At the bottom boundary, a no-slip condition and zero vertical flux of salt are specified. At the top boundary, a free-slip condition and zero vertical flux of salt are specified. These are expressed by:

\[ S = 0, \psi(x) = 1 \cdot 5R(x^2 - x^3/3), \text{ and } \eta(x) = 3R(1 - x) \text{ at } x = \epsilon^{-1} \]  
\[ S_x = 0, \psi_x = 0 \text{ and } \psi_x = 0 \text{ at } x = 0, \]  
\[ S_z = 0, \psi = R \text{ and } \eta = 0 \text{ at } z = 1, \]  
where \( R = T_r/K \) is the non-dimensional river transport. Inclusion of non-zero wind stress at the surface is an easy modification to the model, but is not pursued herein. The remaining boundary conditions to be considered are those at the mouth of the estuary, \( x = 0 \). These are perhaps the most difficult part of the model and will be discussed at some length.

Estuaries empty either into a larger bay or directly onto a continental shelf (see Figure 1). They usually widen abruptly, allowing geometrical and rotational effects to become important. A two-dimensional model is no longer appropriate. To investigate estuarine dynamics in its simplest form, attention must be focussed landward of this outer region. The inshore limit of this region is herein considered to be the mouth of the estuary.

Salinity and velocity distributions at the mouth are functionally dependent upon river flow, depth, horizontal density difference and other parameters. The surface layers become
fresher as the river transport increases. The estuarine circulation becomes stronger for increasing Rayleigh numbers. Consequently internal dynamics determine seaward boundary profiles as well as those within the estuary. The boundary conditions given at the seaward end of the model must be consistent with these internal dynamics. Thus, salinity and velocity profiles cannot be specified as boundary conditions. Preliminary numerical experiments support this result, since unrealistic seaward boundary layers occur where salinity and velocity profiles are specified as seaward boundary conditions. Experimentation also showed that unless a source of salt in the form of a definite salinity value is specified somewhere in the region, the solution $S = 0$ is obtained. Observations suggest that, although the salinity distribution everywhere within estuarine regions is strongly influenced by variation of river discharge and other parameters, the salinity of the deep water near the seaward boundary is least influenced. We have therefore made salinity at the bottom of the seaward boundary invariant, $S_{(0,0)} = 1$, to assure estuarine behavior. In order to obtain the seaward boundary conditions, attention is focused on the dynamics near the estuarine mouth.

In the vicinity of the seaward boundary for the model it is expected that the estuarine circulation is relatively well developed. Pritchard (1954, 1956) has shown that in this situation the salt balance is maintained primarily by a dynamic balance between horizontal advection and vertical diffusion of salt and a vorticity balance is maintained primarily by a balance between buoyancy forces due to horizontal density gradients and vertical diffusion of vorticity. Horizontal diffusion of salt, especially, while shown by Hansen & Rattray (1965) to be essential to the overall estuarine regime, does not appear to be locally important where the gravitational circulation is well developed. In addition, horizontal diffusion of vorticity and horizontal shear in the vertical velocity field are also assumed to be locally unimportant. These conditions are:

\[
S_{xx} = 0, \quad \eta_{xx} = 0, \text{ at } x = 0
\]

and

\[
\psi_{xx} = 0. \quad (5)
\]
Thus, horizontal diffusive fluxes of salt and vorticity are required to be constant, but unspecified, at the open boundary. Although we are unable to provide a completely rigorous justification of these conditions, they do provide a means of completing the mathematical specification of the problem, without inducing boundary layer behavior near the seaward boundary.

**Numerical formulation and procedures**

A finite-difference grid is chosen to be uniform in $x$ and $z$, such that

\[
\begin{align*}
  x_i &= iAx, \quad i = 0, 1, \ldots, I \\
  z_j &= jBz, \quad j = 0, 1, \ldots, J \\
  t^n &= n\Delta t, \quad n = 0, 1, \ldots
\end{align*}
\]

where $Ax = (cI)^{-1}$, $Az = J^{-1}$ and $At$ is the time step whose magnitude depends upon the stability of the differencing scheme that is chosen. The Laplacian operator is approximated by the usual five-point difference scheme. The advection of salt and vorticity, expressed in terms of the Jacobian, $f$, is approximated by using the $f_x$ and $f_z$ forms of Arakawa (1966), respectively. These conserve salinity and salinity squared and vorticity and kinetic energy. Difusion is approximated by the time-centered scheme of DuFort–Frankel (1953). The resulting finite difference analogs to (3) are:

\[
\begin{align*}
  (\eta^o_{ij} - \eta^{n-1}_{ij})/(2At) &= -\eta_{ij}^n(4AxAz) + \sigmaAx^{-1}(\eta_{i+1,j}^n + \eta_{i-1,j}^n - \eta_{i,j+1}^n - \eta_{i,j-1}^n) \\
  &+ \sigmaAz^{-1}(\eta_{i,j+1}^n + \eta_{i,j-1}^n - \eta_{i+1,j}^n - \eta_{i-1,j}^n) \\
  &- \sigmaRa(S_{i,j+1}^n - S_{i,j-1}^n)/(2Ax), \\
  (S_{ij}^o - S_{ij}^{n-1})/(2At) &= -S_{ij}^n(4AxAz) + KAx^{-1}(S_{i+1,j}^n + S_{i-1,j}^n - S_{i,j+1}^n - S_{i,j-1}^n) \\
  &+ Ax^{-1}(\eta_{ij+1}^n + \eta_{ij-1}^n - \eta_{ij+1}^n - \eta_{ij-1}^n). \\
\end{align*}
\]

The streamfunction at the latest time step, $\psi_{ij}^{n+1}$, is calculated by means of a direct solution (Buzbee et al., 1970) to the finite difference Poisson equation,

\[
\eta_{ij}^{n+1} = Ax^{-2}(\psi_{i+1,j}^{n+1} + \psi_{i-1,j}^{n+1} - 2\psi_{i,j}^{n+1}) + Az^{-2}(\psi_{i,j+1}^{n+1} + \psi_{i,j-1}^{n+1} - 2\psi_{i,j}^{n+1}).
\]
energy, salinity and streamfunction fields at two adjacent time steps are less than 1% of the average kinetic energy, salinity and streamfunctions, respectively.

In all of the computations, the choice of a value for the model estuary length, \( L \), or equivalently the aspect ratio, \( e \), depends upon the estuarine dynamics. The dynamical length of the estuary, \( L_d \), is parameter dependent, with \( L_d < L \). For large river flows, as in the Columbia River, the dynamical length is small compared to the geomorphic length classically associated with this estuary, while for low river flows, such as that of the Delaware Estuary, they are likely to be of the same order. The location of the seaward boundary, \( x = o \), is fixed by assigning a bottom salinity value and thus determining \( \Delta S_b \). The location of the river boundary, \( x = e^{-1} \) and \( S = o \), is then adjusted for each experiment to effectively resolve the gravitational circulation, salinity intrusion and stagnation point. Thus, \( L \) is optimized to provide efficient use of the horizontal grid points. If \( L \) is too small, a smooth transition to the freshwater of the river will be prevented, since the upstream boundary will be too close to the mouth. If \( L \) is too large, there will be too few grid points to resolve the dynamics close to the estuary mouth. Initially, \( L \) is estimated at the beginning of a calculation and then is adjusted after a small number of iterations, if necessary.

**Initial conditions**

Optimum initial fields are desired to shorten the time needed to reach a steady-state solution. In most cases the initial conditions were adapted from the similarity solution of Hansen & Rattray (1965)

\[
\begin{align*}
\psi_{ij} &= \psi_s(1 - ex_i) + \psi_r, \\
\eta_{ij} &= \eta_s(1 - ex_i) + \eta_r, \\
S_{ij} &= S_s(1 - \sin \frac{\pi ex_i}{a}),
\end{align*}
\]

where

\[
\begin{align*}
\psi_r &= 1.5R(z_j^3 - z_j^2/3), \\
\psi_s &= \nu R(-2z_j^3 + 5z_j^2 - 3z_j)/48, \\
\eta_r &= 3R(1 - z_j), \\
\eta_s &= \nu R(-4z_j^2 + 5z_j - 1)/8, \\
S_s &= 1 + \nu \left[ \frac{\nu R(-2z_j^3 + 6z_j^4 - 5z_j^3)/240}{R(-z_j^4 + 4z_j^3 - z_j^2)/8} \right].
\end{align*}
\]

and where \( \nu \) is a function of \( Ra, R \) and \( K \).

**Finite difference boundary conditions**

The computational grid is extended one grid point beyond the bottom, top and mouth of the estuary. This allows the finite difference representation of derivatives at the boundary to be consistent with interior computations. Values of \( S, \psi \) and \( \eta \) outside the boundaries are defined as

\[
\begin{align*}
(1) \text{ Bottom } (j = o) \\
S_{i,j}^n &= S_{i,j+1}^n; (S_z = o), \\
\psi_{i,j}^n &= \psi_{i,j+1}^n; (u = o).
(2) \text{ Top } (j = J) \\
S_{i,j}^{n+1} &= S_{i,j-1}^n; (S_z = o), \\
\psi_{i,j}^{n+1} &= 2\psi_{i,j}^n - \psi_{i,j-1}^n; (w = o).
(3) \text{ Mouth } (i = o) \\
S_{i-1,j}^{n+1} &= 2S_{i,j}^n - S_{i+1,j}^n; (S_{xx} = o), \\
\psi_{i-1,j} &= 2\psi_{i,j} - \psi_{i+1,j}; (\psi_{xx} = o), \\
\eta_{i-1,j} &= 2\eta_{i,j} - \eta_{i+1,j}; (\eta_{xx} = o).
\end{align*}
\]
Two-dimensional circulation model

Solutions at the boundaries are obtained by substituting these values into the difference equations, (7), (8) and (9). The bottom vorticity is evaluated using the first order Taylor series approximation developed by Bryan (1963):

$$\eta^{n+1} = 2\psi^{n+1}/\Delta x^2.$$  

At the mouth of the estuary, a three point forward difference scheme for the salinity gradient,

$$S^n_{x1, i=0} = \frac{1}{2\Delta x} (-3q_j + 4q_{i+1} - q_j),$$  

is used in equation (7) to calculate vorticity boundary values. We double integrate the boundary vorticity field by means of a Gaussian-elimination procedure to calculate $$\psi$$ at the seaward end of the estuary. Given $$\psi$$ on all boundaries, we invert Poisson's equation to obtain the interior streamfunctions. Salinity and velocity profiles obtained by this method are slightly closer to the similarity solution than are those obtained by using the central difference representation of the salinity gradient. A third method, in which the boundary vorticity is simply extrapolated out from the interior,

$$\eta^{n+1}_{ij} = 2\eta^{n+1}_{ij} - \eta^{n+1}_{i+1,j},$$  

also produces acceptable solutions.

Simple extrapolation, however, does not work well when calculating the salinity distribution at the mouth. We have found that simple smoothing produces significantly lower values for the salinity throughout the estuary. Smaller horizontal salinity gradients occur and as a consequence lower values of the streamfunction and velocity fields result throughout the interior. The full salinity equation (8) must therefore be used for the calculation at the open boundary.

**Discussion of results**

Steady-state solutions to the model equation were obtained using a 33 x 33-point finite difference grid. Initially, a 17-point vertical resolution seemed adequate; however, tests with similarity solutions indicated that truncation errors may produce significant differences between analytical and numerical values of $$\psi$$ and $$\eta$$.

Results of computations not included here showed that variations of $$A$$ between 1 and $$10^6$$, and $$\sigma$$ from 1 to $$10^2$$ have a negligible effect upon the results. Prandtl number independence is common in thermal convection problems (Beardsley & Festa, 1972). Variation of $$K$$ from 1 to $$10^3$$ does produce significant change in the solutions, but this effect will not be fully explored here.

The model was run for a range of parameters characteristic of, or centered on, nominal values typical of coastal plain estuaries for which data have been published. These nominal values are $$\Delta S_p = 30\%$$, $$K = 10^8$$ ($$K_s = 1 \text{ cm}^2/\text{s}$$), $$Ra = 3 \times 10^6$$ ($$H = 10 \text{ m}$$), and $$R = 2 \times 10^8$$ ($$U_f - 2 \text{ cm/s}$$).

Contours of the streamfunction and salinity distribution obtained are presented in Figures 3 and 8. The salinity distribution shows a stratified intrusion into the estuary, and the streamfunction shows the typically estuarine pattern of seaward flow of near surface water and a landward flow of deeper water.

The vertical component of flow is of considerable interest in connection with estuarine problems, but is not directly, or in many cases indirectly, measurable. It is of interest therefore, to explore the magnitude and structure of the vertical flow associated with the
conditions and parameter ranges of the model. The longitudinal variation of vertical velocity at three levels for \( H = 10 \) m and \( U_i = 2 \) cm/s is shown in Figure 2. The boundary conditions require the vertical flow to be identically zero on the top and bottom boundaries. Both the order of magnitude and the vertical structure of the vertical flow are consistent with the determination of vertical velocity in the James River estuary made by Pritchard (1954). The principal feature of this longitudinal variation is that at all levels a maximum occurs somewhat seaward of the stagnation point (intersection of the internal zero of the streamfunction with the bottom) and a rapid falloff across the position of the stagnation point.

\[ \begin{align*}
\text{Figure 2. Longitudinal variations of the vertical velocity field at } z &= 0.25, 0.5 \text{ and } 0.75 \text{ for } U_i = 2 \text{ cm/s. } Ra = 3 \times 10^6 \text{ (} H = 10 \text{ m), } \sigma = 10, K = 10^6. \text{ Stagnation point is at } x = 60 \text{ km. } - - - , z = 0.50; \cdots - , z = 0.75; \quad - - - , z = 0.25. \\
\end{align*} \]

The model provides a mutually consistent system of salinity, advection and diffusion fields which can be used for investigation of a variety of kinematic problems of biological or geological origin in estuaries. Such applications will require (empirical) determination of the model parameters required to represent a particular estuary, and provide a vehicle for modelling biological or geochemical processes. Our focus here, however, is on the interactions of purely physical processes in estuaries.

**Influence of river discharge**

Effects of river discharge on the estuarine circulation and the salinity distribution are presented for \( Ra = 3 \times 10^6 \) (\( H = 10 \) m). The river transport parameter, \( R \), is varied between \( 5 \times 10^5 \) and \( 6 \times 10^5 \), corresponding to a range in \( U_i \) of \( 0.5 \) to \( 6.0 \) cm/s. Streamfunction and salinity distributions from this series of numerical experiments are shown in Figure 3. Both the stratification and the strength of the estuarine circulation increase with increased freshwater discharge. Vertical profiles of salinity and horizontal velocity, at the seaward end of the modelled region (Figure 4), where the estuarine circulation is well developed, are very similar to the solutions obtained by Hansen & Rattray (1965). The principal response to changes in volume of freshwater discharge occurs in the upper half of the water column. Velocity profiles contain a more or less invariant region near \( z = 0.4 \), below which velocity profiles differ little. Current measurements at this level, or elsewhere in the bottom half of the estuary, would be unable to discriminate between these profiles;
Figure 3. Salinity and streamfunction fields as a function of river flow, $U_f$. Salinity fields are contoured from 0 to 1 in intervals of 0.1. The streamfunction fields are scaled by $10^{-2}$ and contoured from 0 in intervals of 0.4. $Ra = 3 \times 10^8 (H = 10 \text{ m})$, $\sigma = 10$, $K = 10^6 (K_v = 1 \text{ cm}^3/\text{s})$. 
nevertheless, they are quite different overall. The total landward transport into the lower layer is almost independent of the amount of freshwater discharged. This is a somewhat surprising result, but may be due in part to making the bottom salinity at the seaward entrance independent of river flow. Upstream attenuation of the landward flow is considerably more rapid for larger river flows.

This model varies qualitatively from the similarity solutions in that the vertical profiles are not constrained to be similar throughout the estuary. Two features of particular interest are the length of the salinity intrusion into the estuary and the position of the stagnation point. The horizontal variations of salinity at the surface and at the bottom are shown in Figure 5. The longitudinal patterns are similar to the exponential forms given by Hansen & Rattray (1965), except that here we obtain what is not available as an analytic similarity solution: a complete transition to zero salinity. The length of the salinity intrusion, defined by the
distance over which $S > 0.15\%$ at the bottom of the estuary, is a strong function of freshwater discharge. It increases from 44 km for $U_t = 4$ cm/s to 170 km for $U_t = 0.5$ cm/s as shown in Figures 5 and 6. This behavior has been empirically, if qualitatively, known to hydraulic engineers for many years. The functional form of the dependence of salinity intrusion upon freshwater discharge is of special interest because a characteristic value of $U_t$ in coastal plain estuaries is approximately 1 cm/s. It is apparent from Figure 6 that for the values of other parameters used, the length of salinity intrusion changes behavior in the vicinity of $U_t = 1$ cm/s. Increases of $U_t$ from 1 cm/s lead to modest retreat of salinity intrusion, but reductions of $U_t$ result in greatly increased salinity intrusion. The implication of this non-linear relationship for reduction of freshwater discharge into already critical estuaries is fairly obvious. Within the range of values explored, the salinity intrusion varies approximately as $U_t^{-4/7}$ for $U_t > 1$ cm/s and as $U_t^{-5/6}$ for $U_t < 1$ cm/s.

The location of the stagnation point is closely related to the extent of salinity intrusion as may be seen from Figure 3. For $U_t > 1$ cm/s, the position of the stagnation point varies approximately as $U_t^{-5/8}$, and as $U_t^{-5/6}$ for $U_t < 1$ cm/s. Its position is shown in Figure 5 to be bracketed by the (dimensional) salinity values 1.5\% and 0.15\% at the bottom. This result could be of great value to engineers, but it must be understood that the particular values obtained here are a function of the parameters used for these model runs. This fact is demonstrated also in Figure 6 by means of results from model runs using the same set of parameters except for $K$, which was increased by a factor of 5. Establishment of a criterion for the location of the stagnation point based on salinity intrusion, for particular estuaries in which good estimates of the exchange coefficients are obtainable, seems possible in principle. The exchange coefficients of course cannot be those inferred from application of a one-dimensional model to data.

Figure 7 shows contours of the vertical velocity field at mid-depth, $z = 0.5$, as a function of $x$, and $U_t$, for the range of parameters explored. Increasing river discharge results in an increase in the magnitude of the vertical velocity as well as a seaward displacement of its spatial maximum.
Influence of depth

Effects of the depth of an estuary on the estuarine circulation and salinity distribution are presented for a non-dimensional river discharge, $R$, of $2 \times 10^3$. The depth is varied from 7.5 to 12.5 m by varying the Rayleigh number, $Ra$, from $1.3 \times 10^9$ to $5.9 \times 10^9$ (this of course results in a variation of $U_l$, inversely as the depth). Streamfunctions and salinity distributions are shown in Figure 8. Increasing the depth increases the strength of the gravitational circulation and results in greater salt penetration. The landward and seaward flow both increase with increasing depth, but their integral transport is constant and equal to the river discharge. Vertical profiles of salinity and horizontal velocity at the seaward boundary are shown in Figure 9. At the mouth, the speed of the seaward flow near the upper surface is a weak function of the depth of the estuary, but the landward flow is increased except very near the bottom. A current measurement within a meter of the top or the bottom would not discriminate between these profiles, although, to be sure, boundary layer phenomena are not well resolved here. In general, there is a striking difference between the responses of the vertical profiles, at the mouth, to depth and to freshwater discharge. Whereas changes in river discharge affect the horizontal velocity primarily in the top half of the water column, depth changes influence primarily the landward transport in the bottom half of the estuary.

The salinity stratification decreases with increasing depth. This result is attributable to the tendency for the greater horizontal current shear found in the shallower estuary to increase stratification as described by Hansen (1964).

The general effect of changing depth on the extent of salinity intrusion is inverse to that of river discharge. Salinity intrusion increases from 34 km for a depth of 7.5 m to 118 km for a depth of 12.5 m, as shown by Figures 8, 10 and 11. The position of the stagnation point in the circulation also has a similar behavior, as shown by Figures 8 and 11. As might be expected, the position of the stagnation point shows strong Rayleigh number dependence, varying very closely as $H^2$. Diffusive processes weaken the dependence of the salinity intrusion on Rayleigh number however, causing the length of salinity intrusion to vary approximately as $H^{0.2}$.

Figure 12 shows contours of the vertical velocity field at $z = 0.5$ as a function of $x$ and $H$ for the range of parameters explored. Decreasing the depth of an estuary results in an
Figure 8. Salinity and streamfunction fields as a function of estuarine depth, H. Salinity fields are contoured from 0 to 1 in intervals of 0.1. The streamfunction fields are scaled by $10^{-8}$ and contoured from 0 in intervals of 0.4. $R = 2 \times 10^3$, $\sigma = 10$, $K = 10^6$ ($K_x = 1 \text{ cm}^2/\text{s}$).
Figure 9. Variations of (a) velocity and (b) salinity profiles at the seaward boundary, $x = 0$, with depth, $H$: $- - - -$, 12.5 m; $- - - -$, 10 m; $- - - -$, 7.5 m.

Figure 10. Longitudinal variations of surface and bottom salinity as a function of depth, $H$. $H$: $- - - -$, 7.5 m; $- - - -$, 10 m; $- - - -$, 12.5.

Figure 11. Influence of estuarine depth on salinity intrusion and stagnation point location. $- - - -$, $S = 0.05$; $- - - -$, stagnation point; $- - - -$, $S = 0.005$. 
increase in the magnitude of the vertical velocity as well as a seaward displacement of its spatial maximum.

The caveat regarding dependence of particular values upon the choice of values for the exchange coefficients used in the model must also be accepted here, but the general behavior will be unchanged.

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References