BOUNDARY MIXING AND ARRESTED EKMAN LAYERS: ROTATING STRATIFIED FLOW NEAR A SLOPING BOUNDARY

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INTRODUCTION
Motivation
We are concerned here with the behavior of a rotating, stratified fluid near a sloping rigid boundary, with boundary conditions of zero normal buoyancy flux and no slip. Although this is an interesting fluid dynamical problem in its own right, we have been motivated by two major, and at first sight disparate, topics in physical oceanography. The first, known as "boundary mixing", is concerned with how turbulent mixing at the sloping sides of the density-stratified ocean affects the stratification in the interior. The second topic involves the way in which the combination of strati-
Modification and bottom slope may affect the communication into the interior of the no-slip boundary condition at the seafloor.

**BOUNDARY MIXING** Spatially-smoothed temperature, salinity, and other oceanic properties can be mapped with respect to what are loosely called isopycnal surfaces, along which the potential density does not change locally. As no buoyancy forces arise in moving along them, they are preferred pathways for the flow, even if inclined to the horizontal, and so are important reference surfaces. For those isopycnals that outcrop at the sea surface in subtropical regions it is possible that the temperature and salinity, away from the surface mixed layer, are indeed largely determined by advection and stirring along isopycnal surfaces, without any need for diapycnal (cross-isopycnal) mixing. For the abyssal ocean, however, the constant supply of cold bottom water from polar regions must create a slow upwelling (of about 4 m y\(^{-1}\) on average) that is somehow balanced by downward mixing of heat (Munk 1966). This requirement is certainly well documented for some isolated abyssal basins (Whitehead & Worthington 1982, Hogg et al 1982, Saunders 1987, Whitehead 1989) where inflow of cold water has been directly measured and yet the temperature field is assumed not to be changing (Figure 1).

The mixing rate required is typically a few times \(10^{-4} \text{ m}^2 \text{ s}^{-1}\), much larger than the \(\theta(10^{-5}) \text{ m}^2 \text{ s}^{-1}\) or less found from direct measurements of turbulence in the top one kilometer or so of the ocean (Gregg 1987, 1989). Theoretical arguments about the depth dependence of the interior mixing rate will eventually be settled by direct measurements in the abyss. In the meantime, the potential mismatch has revived interest in the suggestion by Munk (1966), pursued by Armi (1978, 1979a,b), that the required mixing occurs not in the ocean interior, but rather in turbulent boundary layers above the sloping bottom.

It was suggested by Armi (1978, 1979a) and Ivey (1987a,b) that the

*Figure 1* Cross-section of cold water entering a basin within which an isotherm maintains a fixed position due to a balance between advection and diffusion of heat.
Effective vertical diffusivity, $K_V$, at a depth $z$ in the ocean, is given by

$$K_V = \frac{K_{Vb} A_b(z, h)}{A(z)}$$  \hspace{1cm} (1)

where $K_{Vb}$ is the vertical diffusivity in the boundary layer of thickness $h$, $A_b(z, h)$ is the horizontal area occupied by the boundary layers at depth $z$, and $A(z)$ is the horizontal area of the ocean interior (Figure 2). Use of this formula with plausible values for $h$ and $K_{Vb}$ made an effective $K_V$ of $10^{-4} \text{ m}^2 \text{s}^{-1}$ seem attainable, but Garrett (1979a,b) pointed out that (1) requires the boundary layer to be as stratified as the interior. This seemed unlikely in the presence of vigorous mixing unless there is rapid exchange between the boundary layer and the ocean interior, as in fact envisaged by Armi (1979a,b) on the basis of observations of detached mixed layers. Nonetheless, most density profiles to the seafloor (e.g. Armi & Millard 1976) showed a region of reduced stratification near the seafloor, so that a reduction factor to $K_V$ from (1) did seem necessary.

Major new insight into the fluid dynamics of boundary mixing was provided by Phillips et al (1986). They showed that the distortion of isopycnals caused by mixing near a sloping boundary would give rise to buoyancy forces that would drive a secondary circulation (Figure 3), and that this would tend to drive dense water upslope and light water downslope, thus enhancing the upslope transport of density and perhaps making boundary mixing more effective than had been recognized.

Garrett (1990), however, pointed out that it is the vertical, rather than upslope, buoyancy flux that matters, and that the advective contribution to this is countergradient, with dense fluid slumping back down and light fluid rising. Thus boundary mixing appears to be less effective than implied even by reducing (1) to allow for reduced stratification in the boundary layer! On the other hand, the secondary circulation continually acts to restore the stratification on which turbulent mixing can act, without the

![Figure 2](image_url)  

*Figure 2* An ocean basin has a bottom boundary layer of thickness $h$. $A_b(z, h)$ is the horizontal area, at depth $z$, within the boundary layer; $A(z)$ is the interior area at depth $z$. 
need for mixed fluid to be ejected from the boundary layer and new, stratified, fluid entrained.

A quantitative, dynamical discussion of these two roles of the secondary circulation—opposing the vertical diffusive flux but restoring the stratification on which the mixing can act—will be presented later in this review. We turn next, though, to our other main motive for considering the behavior of a stratified, rotating fluid near a sloping boundary.

**ARRESTED EKMAN LAYERS** In a quasi-geostrophic flow in the ocean interior, the basic force balance is between a pressure gradient and the Coriolis force. Near the seafloor the current is reduced by friction so that the pressure gradient will be able to drive a cross-isobar flow. This is the basic physics of an Ekman layer. Lateral variations in the current cause variations of the flux in the Ekman layer and force flow into, or out of, the interior. Coriolis forces acting on this flow then reduce the initial currents in the interior, a process known as spin-down (Figure 4). The

![Figure 3](image)

*Figure 3* (i) Vertical density profile. (ii) Corresponding distortion of an isopycnal surface. The arrows indicate the secondary circulation driven by buoyancy forces.

![Figure 4](image)

*Figure 4* Schematic of the interior circulation driven by the convergence and divergence of the Ekman flux beneath a jet. The Coriolis force on the interior circulation decreases the strength of the jet.
vertical extent of the spun-down region is given by a factor \( f/N \) times the horizontal scale of the flow, where \( f \) is the Coriolis frequency and \( N \) is the buoyancy frequency (e.g. Pedlosky 1979).

When the interior flow is above a sloping bottom, however, this fast spin-down process may be disrupted by buoyancy forces (Figure 5). As the imbalance of pressure gradient and Coriolis force drives fluid cross-slope in the bottom Ekman layer, the advection of density gives rise to a buoyancy force which opposes further motion. If a final balance between pressure gradient, Coriolis force and cross-slope buoyancy force is achieved, there is no further Ekman flux, no fluid injected into, or sucked from, the interior, and no further spin-down of the interior flow. In such circumstances the interior flow sees a nearly free-slip boundary condition at the seafloor.

Rhines & MacCready (1989) drew attention to this important physics and estimated the “shut-down time” for a bottom slope \( \tan \theta \) by assuming the cross-slope flow in the Ekman layer to be comparable with the interior along-slope geostrophic flow \( U \). The cross-slope displacement after time \( t \) is then \( Ut \), giving rise to a buoyancy perturbation of magnitude \( N^2 Ut \sin \theta \), where \( N \) is the interior buoyancy frequency. The cross-slope component \( N^2 Ut \sin^2 \theta \) then balances the pressure gradient \( fU \) after a time

\[
\tau_s = S^{-1}f^{-1},
\]

where \( S = N^2 \sin^2 \theta / f^2 \) is the Burger number based on the slope. As we shall review later, this is an underestimate of the true shut-down time as the average cross-slope flow in the Ekman layer is less than \( U \) and buoyancy forces act progressively as the Ekman layer moves cross-slope. Nonetheless, (2) gives the correct order of magnitude estimate of the time for a sloping seafloor to become “slippery”. It may be much less than the

*Figure 5* Schematic of an isopycnal displaced upslope by an Ekman flux until the buoyancy force balances the pressure gradient.
time required for Ekman divergence to spin-down the interior flow, with profound implications for ocean circulation. We return to this issue later but for the moment note that for a bottom slope of 0.01 and with $f = 10^{-4} \text{ s}^{-1}$, then $S = 0.01$ if $N = 10^{-3} \text{ s}^{-1}$ (representative of the abyssal ocean) or $S = 1$ for $N = 10^{-2} \text{ s}^{-1}$ (on the continental shelf). Thus, allowing for smaller and larger slopes, we see that values of $S$ greater than one as well as much less than one, and shut-down times from hours to years, are of oceanographic interest.

**Difficulties**

Boundary mixing is affected by rotation but does not depend on it. In a rotating reference frame, however, the Coriolis force on the cross-slope flow induced by buoyancy will tend to accelerate an along-slope flow and eventually be balanced by divergence of an along-slope stress. We will see this later in terms of the equations of motion, and show how the steady-state along-slope flow outside the boundary layer, in the ocean interior, becomes part of the solution rather than something which can be prescribed independently. We might therefore be concerned about the relevance of steady-state boundary mixing results in a situation where the along-slope interior flow is different from that required by the steady state solution. Conversely, the simple dynamical considerations involved in the arrest of an Ekman layer on a slope suggest that a steady state should be achievable for any along-slope interior flow. This is inconsistent with the steady state solution discussed above, possibly because the simple arrested Ekman layer arguments assume just advection of the density field and ignore its diffusion. In other words, the boundary mixing models ignore time-dependence and imply an along-slope flow that may not match the prescribed one, whereas the time-dependent Ekman layer models tend to ignore buoyancy diffusion and appear to predict a final steady state when it may not be possible.

**Reconciliation**

Both the boundary mixing and arrested Ekman layer theories deal with the behavior of a stratified, rotating flow near a sloping boundary. It should thus be possible to reconcile the two approaches within the context of a single conceptual model. Recent work has shown that this is largely the case. In brief, the answer is actually quite simple, at least for constant coefficients: The solution of the time-dependent problem adjusts to the steady-state "boundary mixing" solution near the slope. The implied along-slope flow in the ocean interior, if different from the imposed one, tends to diffuse into the interior according to a "slow diffusion" equation.
SLOPING BOUNDARIES

(MacCready & Rhines 1991). For the turbulent case there may be more possibilities if the mixing coefficients adjust with time.

Outline

Although our oceanographic motivations have led us to consider the situation where mixing is largely confined near the boundary, we shall first review the steady-state and time-dependent solutions for constant coefficients of viscosity and diffusivity. Following that, we derive some general results for arbitrary profiles of the mixing coefficients, emphasizing the physics and introducing the key dimensionless parameters that govern the problem. Some results of numerical solutions will also be reviewed.

We then examine the relevance of the theories to the ocean, considering briefly the processes that might lead to enhanced mixing near slopes. We also summarize how the local solutions, which ignore cross-slope variations in properties, can be applied in an ocean with variable bottom slope or interior properties.

Many theoretical and observational problems remain; these will be discussed in the final section of the paper.

GOVERNING EQUATIONS

The basic problem to be considered has no variations in the cross-slope direction and so has governing equations

$$\frac{\partial U}{\partial t} - f V = \frac{\partial}{\partial z} \left( \nu \frac{\partial U}{\partial z} \right)$$

$$\frac{\partial V}{\partial t} + f U = - \frac{1}{\rho_0} \frac{\partial \bar{\rho}}{\partial y} + B \sin \theta + \frac{\partial}{\partial z} \left( \nu \frac{\partial V}{\partial z} \right)$$

$$0 = - \frac{1}{\rho_0} \frac{\partial \bar{\rho}}{\partial z} + B \cos \theta - \frac{\partial}{\partial z} (\bar{w}^2)$$

$$\frac{\partial B}{\partial t} + V N^2 \sin \theta = \frac{\partial}{\partial z} \left( \kappa \frac{\partial B}{\partial z} \right)$$

where the coordinates $y, z$ are upslope and bottom-normal with respect to the plane bottom that is inclined at an angle $\theta$ to the horizontal (Figure 6). Here $U(z, t)$ and $V(z, t)$ are the mean flows alongslope and upslope respectively, $B(y, z, t) = -g(\bar{\rho} - \rho_0)/\rho_0$ is the mean buoyancy in terms of the mean density $\bar{\rho}(z, t)$ and a reference density $\rho_0$, and $N^2$ is the (constant) vertical buoyancy gradient in the interior of the fluid away from the slope.
Figure 6 Definition sketch for the coordinate axes and current components above a sloping bottom at an angle $\theta$ to the horizontal.

(The mean is defined here as the average over the turbulent fluctuations, and so assumes a spectral gap between them and the slowly-varying mean flow.) For flat interior isopycnals we must have

$$\frac{\partial B}{\partial z} \to N^2 \cos \theta \quad \text{as } z \to \infty$$

whereas

$$\frac{\partial B}{\partial y} = N^2 \sin \theta \quad \text{for all } z.$$  

The mean pressure $\bar{p}$ is taken with respect to the pressure in a fluid of density $\rho_0$ at rest. Eddy transports of $x$ and $y$ momentum normal to the slope are represented in terms of the same eddy viscosity $v(z, t)$; as this just represents the ratio of momentum flux to mean gradient we are not necessarily assuming a short mixing length, but we do simplify by taking the same coefficient in both $x$ and $y$ directions. The Reynolds stress component $w^2$ has been retained but is independent of $y$. The eddy buoyancy transport normal to the slope is parameterized in terms of an eddy diffusivity $\kappa$, but again this is merely a ratio of eddy flux to mean gradient without requiring any assumption about the nature of the mixing. The Coriolis parameter $f$ is twice the component of the Earth's rotation normal to the bottom.

This formulation omits any consideration at this stage of cross-slope variation or of the role of sloping interior isopycnals. It also ignores the response of the fluid to the Reynolds stresses and upslope buoyancy flux associated with even the inviscid reflection of internal waves from the slope (Wunsch 1971, Ou & Maas 1986), but the problem as formulated permits discussion of much of the key physics associated with mixing processes near the slope.

The governing equations above may be combined into a single equation
A steady state mass balance in the presence of upslope mass flux requires a diffusive flux through the top of a control volume.

for a single variable such as the along-slope flow (e.g. Garrett 1991). In the steady state case we note that it is first convenient to describe the upslope flow $V(z)$ in terms of a streamfunction $\Psi(z)$ by $V = d\Psi/dz$. Then (6) integrates to

$$\Psi N^2 \sin \theta = \kappa \frac{\partial B}{\partial z}$$

if we take $\Psi = 0$ on $z = 0$ where there is no bottom-normal buoyancy flux.

The physics of this is illustrated by considering the control volume in Figure 7: A volume flux $\Psi$ leaves the upslope end of the box less dense than when it entered, and so must have lost mass by diffusion while in transit. Now $\partial B/\partial z \to N^2 \cos \theta$ (the interior stratification) as $z \to \infty$, so that $\Psi \to \kappa_{\infty} \cot \theta$, with $\kappa_{\infty}$ the value of $\kappa(z)$ as $z \to \infty$ (Phillips et al 1986, Thorpe 1987).

In the steady state (3) may also be integrated once to give

$$v \frac{dU}{dz} = -f(\Psi - \kappa_{\infty} \cot \theta).$$

Physically, the Coriolis force on the upslope transport can only be balanced by an along-slope stress.

From (4) and (5), the steady state equation for the vorticity component parallel to the $x$-axis is

$$\frac{d^2}{dz^2} \left( v \frac{d^2 \Psi}{dz^2} \right) + \left( \frac{f^2}{v} + \frac{N^2 \sin^2 \theta}{\kappa} \right) \Psi = N^2 \sin \theta \cos \theta + f^2 \kappa_{\infty} \cot \theta.$$

Mixing tilts the isopycnals close to the boundary, causing a torque that drives upslope or downslope flow and is balanced by viscous forces.

**CONSTANT MIXING COEFFICIENTS**

**Steady State**

If $v, \kappa$ are independent of $z$ the solution of (11) is (Weatherly & Martin 1978, Thorpe 1987)
\[ \Psi = \kappa \cot \theta [1 - e^{-qz} \cos qz + \sin qz] \]  

with

\[ q^4 = \frac{1}{4} \left( \frac{f^2}{v^2} + \frac{N^2 \sin^2 \theta \kappa}{v \nu} \right) = \frac{f^2}{4v^2} \left( 1 + S \sigma \right), \]

where \( S = N^2 \sin^2 \theta / f^2 \) is the slope Burger number and \( \sigma = \nu / \kappa \) is the Prandtl number. The effect of the boundary is thus confined to a distance, of order \( q^{-1} \), that is less by a factor \( (1 + S \sigma)^{-1/4} \) than the Ekman layer thickness above a flat bottom or for no interior stratification.

The net upslope flow \( \kappa \cot \theta \) was originally discussed by Phillips (1970) and Wunsch (1970) for the nonrotating case; the requirement of no buoyancy flux through the slope causes the isopycnals to bend down to meet the slope at right angles (Figure 8), creating buoyancy forces that drive the upslope flow. The need for such a flow is clear from the control volume in Figure 7; the diffusive loss of mass out of the lid of the box can only be balanced by a divergence of the upslope mass transport. Phillips (1970) has shown that in a tilted tube the advective buoyancy flux in the boundary layers on the tilted top and tilted bottom can be much greater than the interior diffusive flux parallel to the axis of the tube.

The associated along-slope flow for the rotating case is given by

\[ U = \frac{f \kappa}{qv} \cot \theta [1 - e^{-qz} \cos qz] \]

if we apply \( U = 0 \) at \( z = 0 \). We note that the along-slope flow outside the boundary layer is in a direction that might be described as "upwelling-favorable" in the Ekman sense, but its value \( \frac{f \kappa (qv)^{-1}}{q^2} \cot \theta \) is part of the solution rather than being arbitrary.

**Time Dependence**

MacCready & Rhines (1991) have investigated the time-dependent response of the bottom boundary layer for constant \( \nu, \kappa \) but with arbitrary
along-slope interior flow. They point out that this sets up an initial Ekman layer in a time of order $f^{-1}$, with a flux that is either upslope or downslope depending on the direction of the interior flow. They also argue that, outside this layer and at times significantly greater than $f^{-1}$, the acceleration and viscous terms in (4) may be neglected, giving a geostrophically balanced along-slope flow. Differentiation normal to the slope and the use of (5) gives

$$\frac{fdU}{dz} = (\frac{\partial B}{\partial z} - N^2 \cos \theta) \sin \theta. \tag{15}$$

This is the "thermal wind" equation connecting the vertical shear of the along-slope current to the horizontal density gradient, which involves $\partial B/\partial z$ due to our coordinate rotation. The basic physics is that the horizontal pressure gradient balancing the Coriolis force changes vertically if the hydrostatic vertical pressure gradient varies due to horizontal differences in density. We see from (15) that changes in $U$ and $B$ must be connected. Thus the viscous term in (3) may be partly balanced by a cross-slope flow $V$, rather than deceleration alone, in order that (6) should produce geostrophically compatible changes in $B$. Similarly, diffusion of $B$ may also induce a cross-slope flow as well as changes in $B$. MacCready & Rhines (1991) show that combining (3), (6), and (15) leads to an equation

$$\frac{\partial U}{\partial t} = v \left( \frac{1}{\sigma + S} \right) \frac{\partial^2 U}{\partial z^2}. \tag{16}$$

for the evolution of the along-slope flow. The same equation also applies to $B$, since $U$ is assumed to be in thermal wind balance. It is remarkable that the complex processes of advection and diffusion in the $y-z$ plane mimic simple one-dimensional diffusion of momentum or buoyancy normal to the boundary. Equation (16), previously applied by Gill (1981), Garrett (1982), and Flierl & Mied (1985) to the spin-down of geostrophic flows in the ocean interior, is termed the "slow diffusion equation" by MacCready & Rhines (1991) since for $\sigma > 1$ it implies slower adjustment of $U$ than would be caused by viscosity alone. Foster (1989) finds an analogous equation for steady, nonlinear flow over topography.

The slow diffusion equation (16) may be used to help predict the behavior of the cross-slope transport (and its associated along-slope boundary stress) over time. Consider the cross-slope transport equation, formed by taking the $z$-integral of (6) and retaining the time dependence

$$\Psi|_\infty = \frac{-1}{N^2 \sin \theta} \int_0^\infty \frac{\partial B}{\partial t} dz + \kappa \cot \theta. \tag{17}$$

While the slow diffusion equation does not apply for $t < f^{-1}$ or for
$z < q^{-1}$, it is generally applicable at later times over most of the, now thickened, boundary layer. Hence (16) may be used to estimate the integral term in (17). Doing this, MacCready & Rhines (1991) find, for an initial value problem, that the transport decays as $t^{-1/2}$ from its starting value, given by standard Ekman theory, toward its steady value, $\kappa \cot \theta$. The decay time scale, or shut-down time, is given by

$$\tau_s = S^{-2} f^{-1} (1/\sigma + S)(1+S)^{-1}.$$  \hspace{1cm} (18)

In the limit $\sigma \to \infty$, $S \ll 1$, this is equal to the original estimate (2). For $\sigma = o(1)$ and small $S$, the shut-down time is much increased over (2), owing to the diffusive thickening of the density boundary layer.

While the estimate (18) is supported by numerical experiments, its analytical derivation relies on the assumption that the along-slope flow is in thermal wind balance—a condition which may be invalid for various choices of the dimensionless parameters. In particular, with strong density diffusion (e.g. $\sigma = 1$) the flow near the boundary appears to be dominated by the steady balance given earlier. Numerical results from MacCready & Rhines (1991) in fact suggest that the steady solution (14) for the along-slope velocity sets the lower boundary condition for the slow diffusion equation, which then acts to bring the interior velocity to this value, as we know must eventually happen. Thus we are left with a fairly consistent picture of the chain of events, with the shut-down process bringing the cross-slope transport toward $\kappa \cot \theta$ over a time $\tau_s$, and the interior velocity being altered to its final value $f\kappa(qv)^{-1}$ by slow diffusion.

**VARIABLE MIXING COEFFICIENTS**

If the mixing is much larger near the slope, as is likely in oceans and lakes, the behavior is different in many ways from that in the situation with constant coefficients.

**General Steady State Results**

The schematic of the distortion of an isopycnal surface by mixing confined near the bottom (Figure 3), compared with that for a constant mixing rate (Figure 8), suggests that the cross-slope secondary circulation will be bidirectional. In fact, as already remarked, the net upslope transport implied by (9) is $\kappa_{\infty} \cot \theta$, or zero if $\kappa \to 0$ as $z \to \infty$.

**BUOYANCY FLUX** The advective upslope buoyancy flux in the boundary layer is $\int_0^\infty BV \, dz$. With $V = d\Psi/dz$ this may be integrated by parts to obtain

$$-\int_0^\infty \kappa (\partial B/\partial z)^2 (N^2 \sin \theta)^{-1} \, dz$$

for $\kappa_{\infty} = 0$ and using (9). Combining
this with the upslope diffusive flux \( \int_0^\infty -\kappa N^2 \sin \theta \, dz \), the total buoyancy flux \( F_B \) may be written

\[
F_B = \int_0^\infty -\kappa \left[ N^2 \sin^2 \theta + \left( \frac{\partial B}{\partial z} \right) \right] \cos^2 \theta \sin \theta \, dz.
\]  

(19)

The second, advective, contribution to \( F_B \) is essentially a "shear dispersion" addition to the first, diffusive, contribution and is dominant unless the average value of \( \partial B / \partial z \) over the region of large \( \kappa \) is less than \( N^2 \sin \theta \).

If we examine the buoyancy flux across a horizontal, instead of bottom-normal, line, we obtain the same total value \( F_B \) in a steady state (Figure 9), but the diffusive flux may be written

\[
F_{\text{diff}} = \int_0^\infty -\kappa \left[ N^2 \sin^2 \theta + \left( \frac{\partial B}{\partial z} \right) \right] \cos^2 \theta \sin \theta \, dz
\]

(20)

\[
= \int_0^\infty -\kappa \left[ N^2 \sin^2 \theta + \left( \frac{\partial B}{\partial z} \right) \right] \cos^2 \theta \sin \theta \, dz
\]

(21)

and the advective flux is

\[
F_{\text{adv}} = \int_0^\infty \kappa N^{-2} (\partial B / \partial z) (N^2 \cos \theta - \partial B / \partial z) \sin \theta \, dz
\]

(22)

Thus, although \( F_{\text{diff}} + F_{\text{adv}} = F_B \) as given by (19), \( F_{\text{adv}} \) is of the opposite sign to \( F_{\text{diff}} \) if the stratification in the boundary layer is reduced below its interior value so that \( \partial B / \partial z < N^2 \cos \theta \). This curious result is easily understood by referring to Figure 3. With respect to a normal to the bottom, there is a tendency for the water moving upslope to be denser than the water moving downslope, thus augmenting the diffusive transport. On the other hand, with respect to a horizontal line the rising water tends to be less dense than

\[\text{Figure 9} \quad \text{In a steady state the vertical buoyancy flux across a horizontal section through the boundary layer is the same as the upslope buoyancy flux across a normal to the slope.}\]
the sinking water (as expected for a buoyancy-driven flow) thus opposing
the diffusive flux.

This reduction of the total flux can also be recognized by comparing
(19) for the total flux to (20) for the vertical diffusive flux; the reduction
factor \( (\partial B/\partial z)(N^2 \cos \theta)^{-1} \) occurs squared rather than linearly! It is con-
venient to define a mixing effectiveness \( I \) as the ratio of \( F_B \) to the buoyancy
flux \( \int_0^\infty -\kappa N^2 \, dz/\sin \theta \) that would occur if the interior stratification
\( \partial B/\partial z = N^2 \cos \theta \) extended all the way to the boundary. This effectiveness
is then the fraction by which a formula such as (1) must be modified to
allow for reduced stratification near the boundary; it may be small.

\section*{Along-Slope Flow}

A useful general result may also be derived for the
along-slope current in the steady state case by combining (9) and (10) to
obtain

\[ dU/dz = \nu^{-1} f \cot \theta [\kappa, -\kappa(\partial B/\partial z)(N^2 \cos \theta)]. \tag{23} \]

The buoyancy gradient \( \partial B/\partial z \) is only constrained to be greater than
\( -N^2 \sin^2 \theta \sec \theta \) to maintain static stability in the boundary layer, but is
more likely to be positive so that, with \( \kappa(z) \) generally much greater than
\( \kappa_\infty \), integrating (23) from \( U = 0 \) at \( z = 0 \) to the top of the boundary layer,
where \( \kappa = \kappa_\infty \) and \( \partial B/\partial z = N^2 \cos \theta \), gives an along-slope flow \( U_\infty \) that is
negative. This downwelling-favorable along-slope flow for the variable
coefficient case is in contrast to the result (14) for the constant coefficient
case. Here also, though, \( U_\infty \) seems to be a part of the solution, rather than
arbitrary.

\section*{Parameter Space}

While (19) gives the total upslope or vertical buoyancy
flux for any profile of \( \kappa(z) \) that tends to zero as \( z \to \infty \), the diffusivity and
buoyancy gradient are not independent, but rather connected through a
solution to (11). Garrett (1990) showed that the solution, including the
mixing effectiveness \( I \), depended on the Burger number \( S \), the Prandtl
number \( \sigma \) and also on the ratio of the distance \( h \) from the boundary over
which significant mixing occurs to the boundary layer thickness, \( q^{-1} \) from
(13), based on the values \( \nu_0 \) and \( \kappa_0 \) of \( \nu(z) \) and \( \kappa(z) \) at \( z = 0 \).

In general we expect the region of greatly reduced stratification to be
\( \mathcal{O}(q^{-1}) \), so that if \( qh \) is significantly less than one, the second, advective,
contribution to (19) is small and the boundary mixing rather ineffective.
On the other hand, for \( qh \gg 1 \) the mixing extends well into the region
which can restratify under the influence of buoyancy forces and is likely
to be effective. We can examine these expectations for specific models.

\section*{Extended Mixing Region}

If \( qh \gg 1 \), Garrett (1991) has argued that the
first term in (11) is important only within a distance of \( \mathcal{O}(q^{-1}) \) from the
boundary; outside this the along-slope flow is in thermal wind balance given by (15) as well as satisfying (23) required by the buoyancy equation. For $\kappa_\infty \to 0$ these imply that, through the mixing region of thickness $h$,

$$\begin{align*}
\frac{\partial B}{\partial z} &= N^2 \cos \theta \frac{S}{S(1 + S)}^{-1} \\
\frac{dU}{dz} &= -f \cot \theta \frac{S}{S(1 + S)}^{-1}.
\end{align*}$$

These formulae apply even if the Prandtl number $\sigma$ is a function of $z$. If $\sigma$ is constant, however, integrating (25) over the mixing region leads to

$$U_\infty \approx -fh \cot \theta \frac{S}{S(1 + S)}^{-1} = -Nh \cot \theta \frac{S^{1/2}}{S(1 + S)}^{-1}$$

which could be $O(1)$ m s$^{-1}$ if $N = 10^{-2}$ s$^{-1}$, $f = 10^{-4}$ s$^{-1}$, and $\tan \theta = 10^{-2}$ so that $S = 1$, if $h$ is $O(10^3)$ m and $\sigma$ is not large.

In this possible flow regime the Richardson number for a small slope is $(\partial B/\partial z)(\partial U/\partial z)^{-2} \approx \sigma(1 + S)\sigma$ which is likely to be greater than 1, implying the need for some mechanism other than shear instability of the flow itself to maintain the mixing.

The mixing effectiveness $I$ for this flow, assuming the first term in (19) to be small, is given approximately by $(S\sigma)^2(1 + S\sigma)^{-2}$ if $\sigma$ is independent of $z$. The cross-slope flow $V$ is upslope close to the boundary, but weak and downslope in the extended mixing region if $\kappa$ decreases with $z$. This solution is of particular interest if $S$ is small, as it shows the possibility of an extended mixing region with weak stratification within which the along-slope flow is nonetheless in geostrophic balance. This contrasts with the situation for $qh \ll 1$ in which $U$, as well as $B$, is affected by the mixing.

**Slab model** A particular situation with $qh \ll 1$ is a slab model in which the mixing is vigorous within a height $h$ above the bottom, and vanishingly small for $z \gg h$. Garrett (1991) showed that care is required in determining and applying the boundary conditions at $z = h$. In particular, he found that the along-slope flow in $z < h$ is very weak, but then jumps to a value

$$U_\infty = -\frac{3}{8} fh \cot \theta \frac{S}{S(1 + S\sigma_2)^{-1}}$$

just outside the slab, in the limit of vanishing viscosity $\nu_2$ and diffusivity $\kappa_2$ for $z \geq h$, where $\sigma_2$ is the Prandtl number $\nu_2/\kappa_2$.

The buoyancy gradient is also very weak in the boundary layer, but has a jump across the top of the slab given by

$$\Delta B = \frac{3}{8} N^2 h \cos \theta \frac{S\sigma_2}{S(1 + S\sigma_2)^{-1}}.$$

This is less than the value $\frac{1}{2}N^2 h \cos \theta$ that would arise from mixing alone,
over a thickness $h$, of an original profile with buoyancy gradient $N^2$. It is thus qualitatively compatible with the downwelling-favorable, negative, value of $U_\infty$ in (27). However, this slab solution for strong mixing not only fails to satisfy a thermal wind balance in the slab but also gives a jump in $U$ across the top of the slab that is of opposite sign to the value $f^{-1}\Delta B\sin \theta$ that would be expected from (15).

For this model of a rather well-mixed slab ($qh \ll 1$), and with $f = 0$ and $v = \kappa$ for simplicity, the second, advective contribution to the upslope buoyancy transport $F_B$ given by (19) is reduced by a factor $2.1 \times 10^{-4}(qh)^8$. This is very small due to the coefficient as well as the high power of $qh$. The calculation depends on assuming constant $v, \kappa$ in the slab, making it rather “stickier” than appropriate for a turbulent flow in which the effective mixing rates are reduced near the boundary. However, in the limit of free-slip boundary conditions the coefficient of $(qh)^8$ only increases by a factor of 4 or so. Thus the advective upslope contribution to $F_B$ may be very small for a model with top-hat profiles of the mixing coefficients, and the purely diffusive transport, given by the first term of (19), may dominate. Salmun & Phillips (1992) have argued that this was the case in some laboratory experiments in which boundary mixing was induced by oscillations of a rough inclined plane.

On the other hand, the factor $\sin^2 \theta$ in the first, diffusive, term in (19) suggests that at small slopes the second, advective, term is likely to dominate the integral if $\kappa$ falls off gradually, permitting a significant $\partial B/\partial z$ to develop in the transition region. This was the case in calculations by Garrett (1990) for exponential profiles of $v, \kappa$ which showed dominance of the advective contribution to $F_B$ even for fairly small values of $qh$.

More numerical integration of the governing equations for other profiles of $v, \kappa$ would be worthwhile. In particular, it would be interesting to see if the abrupt jump to $U_\infty$ given by (27) still occurs if the top-hat mixing profile is smoothed. Such numerical integration, however, would probably best be carried out in the context of the full time-dependent problem.

**Time Dependence**

If the mixing is described by coefficients that vary with distance from the bottom but are independent of time, we might expect that the solution of the time-dependent equations would tend to the steady state solution close to the boundary, followed by slow diffusion into the interior of the downwelling-favorable along-slope flow given as $U_\infty$ in the steady solution. On the other hand, if the fluid motion near a sloping boundary is partly driven by the along-slope flow in the interior, the mixing coefficients may themselves change with time in response to changing conditions of the mean stratification and shear and this could greatly affect the nature
of the solution. In particular, we expect a marked asymmetry between upwelling and downwelling conditions, as seen in the numerical results in Figures 10 and 11.

If the initial Ekman flux is downslope, this is likely to lead to static instability and enhanced mixing. To avoid static instability of the mean density profile the boundary layer will need to continue to thicken as long as

Figure 10  Results from MacCready & Rhines (1992) of a numerical integration of (3)–(6). The diffusivities (σ = 1) varied between 10⁻⁴ and 10⁻² m² s⁻¹ based on the gradient Richardson number. For this run f = 10⁻⁴ s⁻¹, N = 3.5 × 10⁻³ s⁻¹, and sin θ = 0.01 so that S = 0.12. Initially the cross-slope velocity and B' were zero, and the along-slope velocity was downwelling-favorable with magnitude 0.1 m s⁻¹. (These runs used a different sign convention, with positive along-slope velocities being downwelling-favorable.) The velocities (a) and (b) and density anomaly ρ - ρ₀ (c), are plotted at 0.5 days (solid), 3 days (long dashes), and 9 days (short dashes). The integrated cross-slope transport is plotted in (d). Note the decreasing transport and the thickening of the mixed layer by static instability. The along-slope velocity approached a linear shear, nearly in thermal wind balance with the density field.
as there is any downslope flux (although it is also possible that in reality intrusions will peel off into the interior). If the initial Ekman flux is upslope, however, there is no reason for the boundary layer to grow in thickness. It could, in fact, become thinner as the density difference across the top of a mixed layer near the boundary increases due to advection, suppressing the mixing and permitting buoyancy-driven restratification.

This asymmetry was clearly shown in the pioneering study by Weatherly & Martin (1978) which used the Level II turbulence closure model of Mellor & Yamada (1974). Very similar results to this were obtained by Jin (1990) and Trowbridge & Lentz (1991) using a slab model, and by MacCready & Rhines (1992). Field observations (Figure 12) also show a significantly thicker mixed layer when the along-slope flow is downwelling-favorable than when it is upwelling-favorable.
Figure 12  Time series of the along-shelf current 7 m above a sloping bottom off northern California compared with the thickness of a well-mixed bottom boundary layer (from Lentz & Trowbridge 1991). Positive current is poleward and downwelling-favorable.

Trowbridge & Lentz (1991) and MacCready & Rhines (1991) assumed flat interior isopycnals, a steady along-slope velocity $U_e$ outside the boundary layer, and used a perturbation buoyancy

$$B' = B - N^2 y \sin \theta - N^2 z \cos \theta$$  \hfill (29)

to represent the change in buoyancy from the initial stratification. With the acceleration terms in the momentum equation also ignored (on the grounds that they contribute only to irrelevant initial inertial oscillations), the governing equations become

$$-fV = \frac{\partial}{\partial z} \left( v \frac{\partial U}{\partial z} \right)$$  \hfill (30)

$$f(U - U_e) = B' \sin \theta + \frac{\partial}{\partial z} \left( v \frac{\partial V}{\partial z} \right)$$  \hfill (31)
Assuming that the mixing is confined to a layer of thickness \( h \), Trowbridge & Lentz (1991) integrate these equations over the boundary layer and assume that the bottom stress obeys a quadratic drag law based on the average velocity \((\bar{U}, \bar{V})\) over the layer. Hence

\[
- f h \bar{V} = -C_d (\bar{U}^2 + \bar{V}^2)^{1/2} \bar{U}
\]

\[
f h (\bar{U} - U_e) = -C_d (\bar{U}^2 + \bar{V}^2)^{1/2} \bar{V} + B' \sin \theta
\]

\[
dB'/dt = -N^2 h \sin \theta \bar{V},
\]

where \( B' \) is the integrated extra buoyancy in the layer. They determine \( h \) from a closure condition based on the layer Richardson number given by

\[
R_b = g (\Delta \rho / \rho_0) h / (\Delta u)^2 = \frac{-B' + \frac{1}{2} N^2 h^2}{(\bar{U} - U_e)^2 + \bar{V}^2},
\]

where \( \Delta \rho \) is the difference between the average density in the boundary layer and the density just above it; \( \Delta u \) is the difference between the average velocity in the layer and the imposed velocity outside the layer. If \( R_b < 1 \) they increase \( h \) to make it equal to 1, and otherwise leave \( h \) unchanged.

Integration of this simple slab model, which is essentially that of Pollard et al (1973) and is also derived by Jin (1990), gives results, for both upwelling and downwelling situations with particular parameter choices, that are very close to those obtained by Weatherly & Martin (1978) from a more elaborate numerical model.

The time scale for significant reduction in the Ekman flux, based on the initial values of the cross-slope transport and mixed layer thickness, is

\[
t = 2^{-3/4} (C_d N / f)^{-1/2}
\]

here again the fundamental time scale \( S^{-1} f^{-1} \), derived in the introduction, appears. While the model solution evolves towards a steady state, with zero flow in the slab, Trowbridge & Lentz (1991) downplayed this on the grounds that the along-slope velocity component in the boundary layer would develop a thermal wind shear due to the horizontal buoyancy gradient associated with the tilted isopycnals of a mixed boundary layer—particularly for a downwelling-favorable initial flow. In this case, assuming slab-like behavior and expressing the bottom drag in terms of the average velocity over the layer become inappropriate. The thickening of the bottom mixed layer in the downwelling case may also substantially increase the time for arrest; we return to this issue later.

MacCready & Rhines (1992) integrated the full equations numerically,
using a gradient Richardson number formulation for the turbulent mixing; examples of their results are shown in Figures 10 and 11. Guided by their results, they also developed an analytical model which assumes a well-mixed density profile throughout the boundary layer but also allows the velocity to develop significant shear, including the thermal wind shear. For a constant viscosity \( \nu_0 \) in the mixing layer, they integrate (30) and (31) in the vertical to obtain

\[
\frac{dB'}{dt} = \left[ 2f h / (\delta \sin \theta) \right] dM / dt,
\]

where \( M = \int \nu dz = h \tilde{V} \) is the total transport across the slope and \( \delta = (2\nu_0 f)^{1/2} \) is the natural Ekman thickness. Combined with (35) which may be written \( dB'/dt = -N^2 \sin \theta M \), this gives

\[
\frac{dM}{dt} = -\frac{1}{2}(\delta / h) M / \tau_s,
\]

where \( \tau_s = S^{-1} \nu^{-1} \) is the original estimate of the shut-down time in (2).

This analysis shows that the shut-down process is exponential (for constant \( \nu_0 \) and \( h \)), but with a time scale that is increased to \( (2h / \delta) \tau_s \) due to the dilution of the buoyancy force by mixing. It also suggests that downwelling boundary layers, which become thicker, may take longer than upwelling layers to come to a halt. The use of a constant eddy viscosity near the slope in this model makes possible comparisons between theoretical and numerical solutions, but leaves the boundary stress higher than would be produced by a log layer. Also, both Trowbridge & Lentz (1991) and MacCready & Rhines (1992) in their analytical model assume that it is possible at late time in the solution to have a well-mixed buoyancy layer within which the velocity field is in a mainly thermal wind balance; this is hard to reconcile with the steady state solutions described earlier which suggest that for \( qh \ll 1 \) the velocity as well as the buoyancy is well mixed, whereas for \( qh \gg 1 \) the along-slope velocity is in thermal wind balance but the buoyancy gradient normal to the slope is nonzero. Thus the evolution of the time-dependent solution may be hard to describe with simplified models, particularly if \( S \) is not small.

The most remarkable feature of the numerical solutions shown in Figures 10 and 11 is the tendency for thermal wind shear to replace viscous shear. Buoyancy advection tilts the isopycnals in this way, whether the flow is up- or downslope. Classic stratified spin-up carries out this same process, yet in that case the vertical scale of the thermal wind field is the Prandtl penetration scale \( fL/N \), whereas here it is initially confined to the boundary layer, and then slowly diffuses upward. In the downwelling case (Figure 10) weak stratification persists in the boundary layer [though even weaker than given by (24) with \( S = 0.12 \)]. If the boundary layer had no stratification, thermal wind would bring the along-slope velocity \( U_e \) to rest
at the boundary over a vertical scale of $fU_c/N^2 \sin \theta$; this is 82 m for the parameters of Figure 10—not much more than the thickness of the boundary layer after 12 days and suggesting that an almost steady state is close to being achieved. In the upwelling case (Figure 11) the boundary layer becomes thinner as it progresses up the slope due to the suppression of mixing by the increasing density contrast, followed by buoyancy-driven restratification. This leaves behind some thermal wind and there is also some slow diffusion further into the interior.

Existing models have thus revealed the kinds of behavior to be expected in time-dependent problems. More exploration of parameter space would be worthwhile, however, to examine the evolution of flows forced directly by the mixing rather than as a response to external flows, and also to see how the residual stratification in the downwelled boundary layer changes with increasing values of the Burger number $S$.

Much of the work to date has been for the flat interior isopycnals, although Jin’s (1990) slab model, Bird et al’s (1982) extension of the work of Weatherly & Martin (1978) and Garrett’s (1990, 1991) study of the steady state problem have allowed for sloping isopycnals in the ocean interior and a consequent thermal wind there. Future studies will take this further and include, as one limiting case, the situation when the bottom is flat and only the interior isopycnals slope. In this case the bottom boundary layer thickness is that of an Ekman layer, uninfluenced by stratification, and no buoyancy forces arise to arrest the Ekman flow. Nonetheless, steady state solutions may develop near the boundary and then slowly diffuse into the interior.

**Arrest or Spin-Down?**

We now return to the question raised in the introduction of whether an oceanic current above a sloping seafloor will adjust to the no-slip boundary condition there more quickly by Ekman layer arrest, involving just the current close to the bottom, or the usual spin-down process affecting more of the water column. Both time scales require a specification of the drag law at the bottom; if we take this to be quadratic with a drag coefficient $C_d$ the arrest time is of order $(C_d N/f)^{-1/2} S^{-1} f^{-1}$ whereas the spin-down time is of order $C_d^{-1} (H/U)$. Here $U$ is the along-slope current in the interior and $H$ the vertical distance, of order $f/N$ times its horizontal scale, over which it would be spun down.

The ratio of arrest time to spin-down time is $S^{-1} C_d^{1/2} (Nf)^{-1/2} U/H$ and so is critically dependent on the value of $S$. We will consider different situations later. We note that the ratio may be written as $S^{-1} \delta / H$, where $\delta = C_d^{1/2} U(Nf)^{-1/2}$ is the scale of the bottom boundary layer according to the model of Pollard et al (1973) for a flat boundary, and in fact the same
value of the ratio applies in the laminar case with $\delta$ replaced by the Ekman layer thickness. For the downwelling case we have seen how the boundary layer may become much thicker due to static instability and that this may delay its arrest. A very rough estimate of the shut-down time comes from noting that the final thickness of the mixed layer required for thermal wind to bring the interior velocity $U$ to rest is $Uf(N^2 \sin \theta)^{-1}$ if we assume that there is no density jump across the top of the mixed layer. The amount of water that has moved downslope is then $\frac{1}{2}U^2 f^2(N^2 \sin \theta)^{-2}(\tan \theta)^{-1}$. Taking an initial Ekman flux of $C_d U^2/f$ (though this will be reduced as the thermal wind develops) the shut-down time is then of order $\frac{1}{2}C_d^{-1}N^{-1}S^{-3/2}$ (taking $\cos \theta \approx 1$ for small slopes), illustrating great sensitivity to $S$, and implicitly to $N$ which appears as $N^{-4}$. We will derive some specific values for shut-down and spin-down times later; we note that an observational signal of arrest in the ocean would be either zero bottom stress, or, equivalently from integrating (3), no net cross-slope flow.

APPLICATIONS TO THE OCEAN

Regimes

In regions of the ocean where mean along-slope currents are weak, or have been spun down near the bottom by either Ekman layer arrest or by the interior circulation driven by Ekman layer divergence, we might expect the physics of boundary mixing to be dominant with weak net cross-slope flow. We discuss first the significance and influence of this regime in the ocean before considering the three-dimensional nature of ocean circulation which allows for local cross-slope Ekman flux driven by an adjusting along-slope current.

Mixing Processes Near Slopes

It is clear that the effectiveness of boundary mixing depends upon there being mixing processes that extend further from the slope than the distance over which restratification can occur. As found by Phillips et al (1986) and Salmun & Phillips (1992), this is unlikely for the turbulence generated by bottom friction right at the slope, which is likely to be suppressed in the restratified region and so operate only on water that is rather well-mixed. The result is a very small net buoyancy flux.

Of course intermittent mixing events at the seafloor, followed by the restratification that can occur on a slope (and distinguishes the boundary layer there from that over a flat bottom or at the sea surface), will lead to a net buoyancy flux. Garrett (1991) investigated the kinematic consequences of this, finding that, in the simplest case with no net motion up or down the slope during the event, half of the vertical buoyancy flux
achieved in the event is lost in the restratification! Overall, though, he estimated that the process was unlikely to be important globally.

It has been increasingly recognized, however, that the reflection of internal gravity waves from the sloping seafloor can lead to enhanced mixing for a considerable distance from the slope. The dynamical reason for this is that internal wave rays are reflected at the same angle to the vertical, rather than to the normal, in order to conserve the wave frequency (e.g. Phillips 1977). Thus for incidence from deep water (Figure 13) the ray tube is narrowed leading to an increase in wave energy density to conserve the energy flux. This increase is enhanced by a reduction in the group velocity, and the wave shear increases still further due to the increased vertical wavenumber. Incident waves from some directions are correspondingly reduced, but, allowing for a full incident spectrum, Eriksen (1982, 1985) and Garrett & Gilbert (1988) show that there is a considerable net increase in shear and hence in the probability of shear instability and mixing in the stratified water near the boundary. The latter authors attempted to quantify the amount of energy available for mixing by determining the wavenumber such that the shear spectrum for lower wavenumbers gives a Richardson number of order one, and then arguing that waves with higher wavenumbers would break and give up their energy flux to mixing. Their results were sensitive to the ratio $N/f$ and to the bottom slope but suggested that this form of near boundary mixing could well be significant in the deep ocean, particularly at low latitudes. Ivey & Nokes (1989) have shown in the laboratory that internal waves reflected from a sloping boundary can lead to vigorous mixing in a turbulent boundary layer. Evidence in the ocean for the enhancement of internal waves on reflection has been found by Eriksen (1982), though puzzles in interpreting available current meter data remain (Gilbert 1991). Moreover, following the theoretical analyses of Baines (1971), Gilbert & Garrett

![Figure 13](image-url)

*Figure 13* Internal wave rays incident on a sloping bottom are reflected at equal angles to the vertical.
(1989) argue that the mixing may be less above topography that is concave rather than convex.

More recently, fine-scale and microscale velocity data above the sloping sides of Fieberling Guyot, a seamount in the northeast Pacific Ocean, have shown evidence for significantly enhanced mixing up to 200 m or so above the bottom, well into the stratified water above a well-mixed boundary layer (J. Toole, R. Schmitt & K. Polzin, personal communication; Figure 14), and possibly associated with internal wave reflection. In another energetic flow regime, enhanced turbulence outside a bottom mixed layer has been found in the Florida Straits (D. Winkel & M. Gregg, personal communication; Figure 15).

![Fieberling Guyot Flank Site](image)

**Figure 14** Vertical profiles, above the sloping sides of Fieberling Guyot, of (left) the variance of fine-scale shear (3 to 128 m wavelength), normalized by the local $N^2$, (center) the turbulent kinetic energy dissipation rate $\varepsilon$ averaged over 10 m in the vertical, and (right) the corresponding vertical eddy diffusivity given by $\Gamma\varepsilon/N^2$ (Osborn 1980) using $\Gamma = 0.25$ (J. Toole, R. Schmitt & K. Polzin, personal communication). The turbulence is enhanced near the bottom, yet is surprisingly strong in the upper kilometer as well. (Typical mid-ocean values for $\varepsilon$ correspond to the lowest values seen in this profile.) Internal wave enhancement by the seamount may be responsible for the large shear seen in the lefthand profile; a value of 2 corresponds to a Richardson number of $\frac{4}{3}$. 
A cross-section of turbulent kinetic energy dissipation measured with a free-fall profiling instrument by D. Winkel & M. C. Gregg (personal communication). The site is the Florida Straits at 27°N, where the Florida Current, with speeds of typically 0.5 m s⁻¹, develops intense turbulence near the solid boundary as well as sporadically in the interior. The thickness of the actively mixing bottom layer (solid-dashed curve) often exceeds the thickness of the density-mixed layer (dashed curve). The profiles are nondimensionalized as \( \frac{\langle \epsilon \rangle}{vN^2} \), where \( \epsilon \) is the turbulent dissipation. Even without this nondimensionalization similar features appear.

While internal waves are an important mechanism for near boundary mixing, Thorpe's (1987) measurements on the abyssal continental slope southwest of Ireland showed larger signals at the period of the semidiurnal lunar tide; the generation of internal tides in the ocean may yet turn out to be the dominant mixing process near slopes.

**Patching Local Solutions to the Interior**

The discussion of boundary mixing earlier in this review assumed a constant interior stratification, a uniform bottom slope, and uniform mixing rates so that the whole problem becomes independent of the upslope coordinate. In practice all of these assumptions could be violated. Phillips
et al (1986) considered the effect of boundary mixing on an interior pycnocline, or region of large vertical density gradient. They argued that the enhanced mixing at the slope leads to a spread of the isopycnals up and down the slope (Figure 16) and that the resulting buoyancy forces on this drive a tertiary flow (on top of the secondary circulation associated with the local boundary mixing). This tertiary flow converges at the pycnocline, causing an outflow into the ocean interior and spreading the isopycnals in the same way as if the mixing had occurred in the fluid interior in the first place.

Boundary mixing concepts and results for constant interior $N$ can presumably still be applied locally provided that the tertiary flow is considerably weaker than the secondary flow. Garrett (1991) showed that this is the case if the interior pycnocline thickness is much greater than the thickness of the boundary layer, as is likely to be the case in most geophysical situations. While a formal expansion of the overall solution in terms of a small ratio of boundary layer thickness to vertical height of environmental variability can be pursued, or new boundary layer scalings considered if the interior pycnocline is very thin (Salmun et al 1991), the nature of the solution was described by McDougall (1989). He assumed that the tertiary circulation maintains flat isopycnals (after averaging across the distortion of the boundary layer) and that the interior stratification changes slowly, if at all. Conservation of density in the boundary layer at depth $z$ then requires an average vertical velocity $w_b$ in it given by

$$A_b w_b N^2 = d(A_b K_{eff} N^2)/dz,$$

(39)

where $A_b$ is the horizontal area of the boundary layer at depth $z$ and $K_{eff} N^2$ is the average vertical buoyancy flux per unit area, with $K_{eff}$ reduced below the actual eddy diffusivity there by the effectiveness factor of boundary mixing discussed earlier. Vertical variations of $A_b w_b$ times the length $C$ of

![Figure 16](image)

*Figure 16* If an interior pycnocline is spread across bottom contours by boundary mixing, the convergent buoyancy-driven tertiary circulation in the boundary layer drives a flow into the interior and spreads the isopycnals there.
a depth contour then require net exchange between the boundary layer and the ocean interior given by

\[ C_u = \frac{d(A_b w_b)}{dz} = \frac{d}{dz} \left[ N^2 \frac{d}{dz} \left( A_b K_{\text{eff}} N^2 \right) \right] \]  

(40)

showing that exchange between the boundary layer and the interior may be driven by vertical changes in \( K_{\text{eff}} \) or \( A_b \) as well as \( N^2 \). As pointed out by McDougall (1989), this result implies inflow at the top of a seamount to supply a net flow down its sloping sides. The inflow rate, of course, depends on the strength and effectiveness of boundary mixing.

If the vertical diffusivity \( K_{\infty} \) in the ocean interior is finite, consideration of the tertiary circulation may have to include the net upslope flow \( K_{\infty} \cot \theta \) (Woods 1991), but this is likely to be smaller than the tertiary flow induced by the turbulent boundary mixing. This discussion has also averaged over variations along a particular depth contour and so ignores the way that the tertiary flow might leave the boundary layer at some point to return further along on the same contour. Moreover, the above discussion of the tertiary flow ignores the way in which there may be vigorous lateral exchange between the boundary layer due to flow separation (Armi 1978) or just the turbulent eddies shed into the interior due to lateral instability of an along-slope current, much as in a river interacting with its bounding bed. This is certainly a likely process in energetic flows like the Florida Current (Figure 15), and in the deep ocean is suggested by measurements of chemical tracers and suspended sediments (Eittreim et al 1975). However, the possible overall constraint on the effectiveness of boundary mixing, due to reduced stratification in the mixing region close to the boundary, remains.

The above discussion has also assumed flat interior isopycnals. If this is not the case, as for an along-slope current that has been spun down, with tilted isopycnals and thermal wind shear near the boundary, further investigation of the effectiveness of boundary mixing may be necessary. The basic question, however, is whether a strong along-slope mean flow has been spun down.

**Evolving Along-Slope Currents**

The physics of the arrested Ekman layer might show up in the ocean in response to time varying interior flows (Figure 12). Perhaps more importantly, it might play a role in the dynamics of deep western boundary currents, such as that in the North Atlantic (Figure 17). Here the mean current averages about 0.2 m s\(^{-1}\) near the bottom, with pulses to twice that value, and is downwelling-favorable. It decreases upward with an \( e\)-
Figure 17  Profiles of salinity $S$, potential temperature $\theta$, and turbidity at the Blake-Bahama Outer Ridge (30.3°N, 74.7°W) (Jenkins & Rhines 1980). Here the Deep Western Boundary Current, with average speed at two nearby moorings of 0.21 m s$^{-1}$ and 0.13 m s$^{-1}$ (200 m above bottom), winds southward from sources in the Greenland Sea, along steeply sloping topography (slope $\approx 2 \times 10^{-2}$). The mixed layer of this downslope-favorable current has reached a thickness of 270 m, though more typically it ranges from 40 to 100 m in this region. There are signs of mixing, and intrusion from the boundary, above the bottom mixed layer. (The turbidity records light attenuation due to fine sediment carried from the boundary into the interior.)

folding scale of about 800 m; the shear is in thermal wind balance with a tilt of the density surfaces that is upward towards the slope, holding cold water at an unusual height. Small suspended particles taken up from the bottom appear in the turbidity profile, giving direct evidence of mixing through the density mixed layer and also suggesting a history of older mixing events intruding into the fluid above.

The parameters $N = 1.2 \times 10^{-3}$ s$^{-1}$, $f = 7.3 \times 10^{-5}$ s$^{-1}$, and $\sin \theta = 2 \times 10^{-2}$ here give a Burger number $S = 0.11$. With $C_d \approx 2.5 \times 10^{-3}$
the arrest time \((C_d N/f)^{-1/2} S^{-1} f^{-1}\) for an upwelling situation would be 7 days, whereas the more appropriate downwelling formula \(\frac{1}{2} C_d^{-1} N^{-1} S^{-3/2}\) gives 54 days. Spin-down via Ekman suction would involve most of the water column and have a time scale \(C_d^{-1}(H/U) \approx 70\) days if we take \(H = 3000\) m (as for a current width of about 50 km) and \(U = 0.2\) m s\(^{-1}\). Thus, shut-down via arrest of the Ekman layer would seem to be at least as important as spin-down and it would seem surprising that in the presence of both the bottom velocity still persists. The resolution seems to be related to the upward tilt of the interior isopycnals with a slope almost equal to that of the bottom. In such circumstances little shut-down occurs and the current is regenerated against spin-down by the available potential energy of the interior density field (P. MacCready, submitted for publication).

On significantly smaller slopes in the deep ocean \(S\) would be much less, with spin-down becoming more important than shut-down even without the complication of sloping interior isopycnals. On the continental shelf, however, it is likely that shut-down occurs more rapidly than spin-down, but the ratio of time scales needs evaluation using the appropriate parameter values for each situation (see Trowbridge & Lentz 1991).

OUTSTANDING PROBLEMS

Theoretical Questions

Much has been learnt in recent years about the basic fluid dynamics of mixing near a sloping boundary. In particular, it is clear that the effectiveness of the process depends on whether the mixing mechanism can extend far enough from the boundary that buoyancy-driven flows can continually restore the stratification on which the mixing can act. It is this ability of the boundary layer to restratify that distinguishes it from boundary layers at the surface or above a flat bottom (unless there is an interior thermal wind in those cases). While flow separation is not necessary to bring fresh, stratified water into contact with the boundary, the role of exchange by lateral eddies requires further investigation.

For steady conditions with flat interior isopycnals the nature of the solutions is reasonably clear, and for time-dependent problems we do have a framework which recognizes the tendency for buoyancy forces to arrest an Ekman layer and for the along-slope flow at the edge of the layer to slowly diffuse into the ocean interior. We do not, however, have a complete quantitative understanding of the time for arrest to occur. Further work, clearly described in terms of the independent dimensionless parameters of the problem, is required. The situation becomes more complicated if, as is very likely, the mixing parameters evolve in response to changing conditions of flow and stratification. The model of MacCready & Rhines
(1992) points to the gradual replacement of viscous shear with thermal wind shear, but the stratification in the final state is rather uncertain.

Allowing for sloping interior isopycnals adds an extra richness to the problem. As discussed earlier, this has been included in some studies, but more investigation is required, with interesting possible applications to the particular example of an interior front near a flat surface.

We have argued that patching local solutions to the interior flow is straightforward if the vertical scale of parameter variation is greater than the boundary layer thickness. The effect of the Earth's rotation on the tertiary flow remains to be investigated.

Observational Needs

The theoretical discussion has largely depended on representing eddy fluxes of buoyancy and momentum in terms of mixing coefficients. We have pointed out that this need not imply that mixing length arguments are valid, but it is clear that future representations of the eddy fluxes will have to be guided by their measurement in the field. This presents considerable technical difficulties and also statistical concerns due to the small correlation coefficients expected in an environment dominated by internal waves; these are discussed by J. J. M. van Haren et al (submitted for publication) who present the results of a pilot experiment. This study, and the earlier work of Thorpe (1987) and Thorpe et al (1990), show that slopes can be regions of highly variable motion with overturns and other evidence for mixing occurring to a considerable height above the bottom.

A major concern, of course, is whether the mixing is associated with a negligible net upslope flow, as in steady-state boundary mixing theories, or whether there is still a net Ekman flux, upslope or downslope, due to the ocean interior currents. Instrument inaccuracies make it difficult in practice to establish the cross-slope component of the boundary layer flow even with good coverage (Trowbridge & Lentz 1991). There is also the need to discriminate clearly between Eulerian and Lagrangian mean flows (Ou & Maas 1986). Investigation of the spread of boundary-mixed fluid into the ocean interior, whether by the tertiary circulation discussed here or as part of flow separation and eddy activity, would also be useful.

This review has concentrated on the basic fluid dynamical processes that can be expected to occur at a sloping boundary of a rotating stratified fluid. The topic continues to present interesting problems of considerable importance for models of ocean circulation.

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Literature Cited


