JOURNAL OF COMPUTATIONAL PHYSICS **41**, 115–135 (1981)

# Approximation of Radiation Boundary Conditions

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Received July 15, 1980

A survey of methods for imposition of radiation boundary conditions in numerical schemes is presented. Combinations of absorbing boundary conditions with damping (in particular, sponge filters) and with wave-speed modification are shown to offer significant improvements over earlier methods.

## 1. INTRODUCTION

Radiation boundary conditions appear in a wide variety of physical problems involving wave propagation [1, 2]. For these problems, boundary conditions should be specified for dynamical quantities propagating on characteristics entering the domain of integration, while no boundary conditions are necessary for quantities propagating on characteristics leaving the domain. If the domain of integration is finite, numerical solutions of such a problem may be effected by standard techniques, such as the finite difference method. On the other hand, if the desired region of integration is infinite, numerical solution encounters difficulties because of the necessary limitation of the computational domain to a finite region.

With initial-value or radiation problems, the appropriate boundary conditions at infinity are radiation boundary conditions; that is, the amplitude of waves entering from infinity is required to be zero, while no conditions are placed on outgoing waves propagating to infinity. With scattering problems, the amplitude of an incoming wave from infinity is specified and the amplitudes of the scattered outgoing waves emanating from the physical domain are sought; scattering problems can be easily reformulated as radiation problems. In this paper we discuss the approximation of wave propagation problems in infinite regions by finite discrete problems.

A prototype radiation problem is given by the $n$-dimensional wave equation

$$u_{tt} = \nabla^2 u + f \tag{1.1}$$



0021-9991/81/050115-21$02.00/0

in which $f(\mathbf{x}, t)$ has compact support in $\mathbf{x}$ and the boundary conditions are that there be no incoming waves from $\infty$. In one dimension, the general solution to (1.1) in regions where $f = 0$ has the form

$$u(x, t) = F(x - t) + G(x + t). \tag{1.2}$$

If $f = 0$ for $|x| > X > 0$, then the radiation condition is simply that

$$\begin{aligned} G &= 0 \qquad (x > X), \\ F &= 0 \qquad (x < -X). \end{aligned} \tag{1.3}$$

It is easy to formulate suitable boundary conditions on a finite domain to simulate the radiation boundary conditions (1.3) for the one-dimensional wave equation. The boundary conditions

$$\frac{\partial u}{\partial t} \pm \frac{\partial u}{\partial x} = 0 \qquad (x = \pm X) \tag{1.4}$$

for all $t$ preclude the reflection of waves from $x = \pm X$, respectively. Boundary conditions like (1.4) to simulate (1.3) will be called absorbing boundary conditions to distinguish them from the exact radiation boundary conditions (1.3) which hold asymptotically as $X \to \infty$.

The formulation of suitable absorbing boundary conditions in more than one space dimension is more difficult. In three dimensions, the analog of (1.4) is, in spherical coordinates,

$$\frac{\partial}{\partial t}(ru) + \frac{\partial}{\partial r}(ru) = 0 \qquad (r = R), \tag{1.5}$$

but this boundary condition is exact only for spherically symmetric waves, which are characterized by the property that they are incident normal to the boundary $r = R$. Waves that are not spherically symmetric may still be outgoing, but they only satisfy (1.5) asymptotically as $R \to \infty$, giving an error (reflected waves), due to the finite boundary, that decays only like $1/R^2$ as $R \to \infty$.

For other problems, even the formulation of leading order absorbing boundary conditions may not be immediately obvious. For example, consider the radiation condition for the Schrödinger equation

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2} + f(x, t). \tag{1.6}$$

The solution is

$$u(x, t) = \frac{i}{2}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\omega^{-1/2}\int_{-\infty}^{\infty} dy\, f(y, \omega)\, e^{-\sqrt{\omega}|x-y|}\, dy, \tag{1.7}$$

where $f(y, \omega) = (1/2\pi) \int_{\infty}^{\infty} f(y, t)\, e^{-i\omega t}\, dt$ and $\sqrt{\omega}$ is positive for $\omega > 0$ and $\sqrt{\omega} = -i\sqrt{|\omega|}$ for $\omega < 0$. The radiation condition is imposed through the choice of sign of $\sqrt{\omega}$. As $x \to +\infty$, only waves of the form $\exp(ikx - ik^2 t)$ with $k > 0$ persist, while as $x \to -\infty$, only waves of the form $\exp(ikx - ik^2 t)$ with $k < 0$ persist. The proper analog of (1.4) or (1.5) is not clear, at least to us.

Several methods for imposition of radiation boundary conditions may be considered. One possibility is to map the infinite spatial domain of the radiation problem onto a finite domain and solve the resulting transformed problem on the finite domain. It has been shown [3] that mapping by itself is doomed to fail for radiation problems in which the desired solution oscillates to $\infty$. However, the mapping technique can be useful if the fast variation of the solution is factored off and only the non-oscillatory corrections are sought. In some problems, the latter approach may be viable but in many cases not even the oscillatory behavior of the solution can be determined without resort to numerical methods. Mapping will not be discussed further here.

In Section 2 we survey the use of absorbing boundary conditions to impose radiation boundary kconditions. In Section 3, the use of global dampling to achieve radiation boundary conditions is investigated, and then, in Section 4, the use of damping (sponge) layers is studied. In Section 4, the use of damping (sponge) layers is studied. In Section 5, we combine the methods of Sections 2–4 to obtain improved results. In Section 6, we introduce "sponge filters" that further improve the methods developed earlier. Finally, some representative time-dependent calculations are presented in Section 7.

## 2. Absorbing Boundary Conditions

In one space dimension, the absorbing boundary conditions (1.4) give radiation boundary conditions for the wave equation (1.1). In two space dimensions, the formulation of suitable absorbing boundary conditions is more involved [4–10].

Consider the solution to (1.1) in two dimensions with radiation boundary conditions. In order to find absorbing boundary conditions that are to be applied at a point $x = X$, where $f = 0$, (1.1) is rewritten as

$$\left(\frac{\partial^2}{\partial x^2} - L^2\right) u = 0, \tag{2.1}$$

where

$$L^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial y^2}. \tag{2.2}$$

The two-dimensional analog of the absorbing boundary condition (1.4) is

$$\left(\frac{\partial}{\partial x} + L\right) u = 0 \qquad \text{at} \quad x = X. \tag{2.3}$$

Here $L$ is one of the "square roots" (in the sense of pseudodifferential operators [6]) of the operator $L^2$.

Approximations to $L$ are obtained by expansion in powers of the transverse wavenumber (a long-wave expansion). For plane waves of the form $u(x, y, t) = u(x)e^{i\omega t + ily}$, it follows that

$$L^2 u = (l^2 - \omega^2)u. \tag{2.4}$$

The long-wave expansion of $L$ is obtained by expanding $\sqrt{l^2 - \omega^2}$ in powers of $l/\omega$:

$$\begin{aligned} L &= \pm i\omega\sqrt{1 - l^2/\omega^2} \\ &= \pm i(\omega - l^2/2\omega) + O(l^4). \end{aligned} \tag{2.5}$$

Identifying $i\omega$ with $\partial/\partial t$ and $il$ with $\partial/\partial y$ in (2.4) gives the first-order absorbing boundary condition [4–6]

$$u_{xt} + u_{tt} - \tfrac{1}{2}u_{yy} = 0 \qquad \text{at} \quad x = X > 0, \tag{2.6}$$

where an additional time derivative of (2.3) has been taken.

Higher-order absorbing boundary conditions can be similarly obtained. For example, the next higher-order absorbing boundary condition for (2.1) is [6]

$$u_{xtt} + u_{ttt} + \tfrac{1}{4}u_{xyy} - \tfrac{3}{4}u_{tyy} = 0 \qquad \text{at} \quad x = X > 0. \tag{2.7}$$

There are several problems associated with absorbing boundary conditions of this kind. First, high-order accurate absorbing boundary conditions are quite complicated and may lead to difficult numerical and analytical problems [6].

Second, absorbing boundary conditions are accurate only for waves that are close to normal incidence at the boundary. The transverse long-wave expansion requires that the transverse wavelength be large compared to the normal wavelength. For (2.1), problems arise when $l^2 \gtrsim \omega^2$, i.e., for long or evanescent waves in the $x$-direction. In this case, expansion (2.5) breaks down. Another example of the difficulty is given by the Schrödinger equations (1.6) in which long wavelenths result from low frequency modes. Here

$$L^2 = -i\frac{\partial}{\partial t} \tag{2.8}$$

so the absorbing boundary condition (2.3) requires an approximation to the square root of $\partial/\partial t$. If it is known that the solution $u$ consists only of waves with frequency near the reference frequency $\omega_0$ then $L$ may be approximated by

$$\begin{aligned} L &= \sqrt{\left(-i\frac{\partial}{\partial t} - \omega_0\right) + \omega_0} \\ &\simeq \frac{1}{2}\sqrt{\omega_0} - \frac{i}{2\sqrt{\omega_0}}\frac{\partial}{\partial t} \end{aligned} \tag{2.9}$$

in the radiation condition (2.3). On the other hand, if the solution is broadband including long waves, this approach is bootless.

Third, absorbing boundary conditions are not robust. Since they are applied only at the boundary, any deviation from ideal conditions there will cause reflections that propagate into the computational domain and cannot be easily controlled.

In the remainder of tis paper, we use (2.1) as a model to test various ways to impose radiation boundary conditions. Here $L^2$ can have various forms; Fourier analysis of $u$ in time and all directions except $x$ (normal to the artificial boundary) shows that we can consider $L^2$ to be a scalar in our model tests. Although the model reduces to an ordinary differential equation it has nontrivial interpretations and consequences to be discussed later.

## 3. Wave Damping

An intuitively attractive method for imposition of radiation boundary conditions may be based on Sommerfeld's method for analytical solution of radiation problems [1, 2]. Sommerfeld argued that if a radiation problem is modified slightly by the imposition of a small damping term then the desired radiation boundary conditions will be achieved by solving the problem subject only to boundedness conditions at $\infty$ upon taking the limit of zero damping. For example, we could modify the Schrödinger equation (1.6) by adding a "viscous" dissipation term $\nu u_{xx}$ as in

$$\partial u/\partial t = (i + \nu)u_{xx} + f(x, t) \tag{3.1}$$

for some $\nu > 0$. Then solution (1.7) is still valid with $\sqrt{\omega}$ replaced by $\sqrt{\omega/(1 - i\nu)}$. In order that the resulting solution be bounded at $\infty$, the branch of the square root must be chosen so that $\sqrt{\omega/(1 - i\nu)} \to +\sqrt{\omega}$ for $\omega > 0$ as $\nu \to 0+$ and $\sqrt{\omega/(1 - i\nu)} \to -i\sqrt{|\omega|}$ for $\omega < 0$ as $\nu \to 0+$, reestablishing the radiation boundary conditions as $\nu \to 0+$.

This procedure of adding a small dissipative term allows the simulation of radiation problems in finite regions. Unfortunately the simplest implementation of this procedure, as embodied in Sommerfeld's original prescription, suffers from two important practical defects. First, in steady state radiation problems, dissipation may, if not properly implemented, be ineffective in damping incoming waves, leading to excessive spatial resolution and computational requirements. Second, in time-dependent radiation problems, improper application of dissipation may lead to "ringing" modes that persist for long times and lead to improper representation of transient effects. Both these defects are illustrated by application of damping to the one-dimensional wave equation

$$u_{tt} = u_{xx}. \tag{3.2}$$

Let us consider the radiation problem for (3.2) on the semi-infinite interval $0 \leqslant x < \infty$ with the boundary condition

$$u(0, t) = a \cos kt; \tag{3.3}$$

An exact solution of this radiation problem is the steady oscillation $u(x, t) = a \cos k(t - x)$.

An easy way to impose damping on (3.2) is first to rewrite it as the system

$$v_t = w_x, \qquad w_t = v_x,$$

where $v = u_t$, $w = u_x$. A suitable damped system is

$$\begin{aligned} v_t &= w_x + \mu(x) v_{xx} - \nu(x) v, \\ w_t &= v_x, \end{aligned} \tag{3.4}$$

where for later convenience we allow $\mu(x) \geqslant 0$ and $\nu(x) \geqslant 0$ to depend on $x$. Here $\mu(x)$ is a linear viscous damping coefficient while $\nu(x)$ is a linear "Newtonian cooling" or "friction" coefficient. Let us consider the solution to (3.4) on the truncated interval $0 \leqslant x \leqslant L$ for some large $L$ (to be chosen later) with the boundary conditions

$$v(0, t) = \sin kt, \tag{3.5}$$

$$v(L, t) = 0. \tag{3.6}$$

Here (3.5) follows if $a = -1/k$ in (3.3), while (3.6) is imposed to simulate the lack of knowledge of absorbing boundary conditions like (1.4).

If $\mu(x) \equiv \mu$ is a constant and $\nu(x) \equiv 0$ the steady oscillatory solution to (3.4)–(3.6) is

$$v(x, t) = \operatorname{Im} e^{ikt} \left[ \frac{\sin \alpha(L - x)}{\sin \alpha L} \right], \tag{3.7}$$

where

$$\alpha = k/\sqrt{1 + ik\mu}. \tag{3.8}$$

The maximum error in this solution at $x$ as a function of $t$ is

$$\varepsilon(x) = \left| \frac{\sin \alpha(L - x)}{\sin \alpha L} - e^{-ikx} \right|. \tag{3.9}$$

Several limiting cases of (3.9) are of interest:

(i) If $\mu k \ll 1$, $\mu k^2 L \ll 1$ as $\mu \to 0+$, then $\alpha \sim k - \frac{1}{2} i \mu k^2$ so that

$$v(x, t) \sim \sin kt \frac{\sin k(L - x)}{\sin kL},$$

which is the same standing wave obtained by solution of (3.4)–(3.6) with $\mu = \nu = 0$. In this limit, the effect of damping is negligible and radiation boundary conditions are *not* achieved.

(ii) If $\mu k \gg 1$ as $\mu \to 0+$, then $\alpha \sim (k/2\mu)^{1/2}(1-i)$ so that the waves are distorted by the damping. If $\mu k^{-1}L^{-2} \ll 1$, then

$$v(x,t) = \exp\left(-\left(\frac{1}{2}\frac{k}{\mu}\right)^{1/2} x\right) \sin\left[kt + \left(\frac{1}{2}\frac{k}{\mu}\right)^{1/2} x\right].$$

In this case too, radiation boundary conditions are not obtained.

(iii) If $\mu k \ll 1$ and $\mu k^2 L \gg 1$ as $\mu \to 0+$, then the maximum error (3.9) at $x$ behaves like

$$\varepsilon(x) \sim 1 - \exp(-\tfrac{1}{2}\mu k^2 x). \tag{3.10}$$

Thus, accurate simulation of the outgoing wave can be achieved on a fixed subinterval $0 \leqslant x \leqslant x_0$ of $0 \leqslant x \leqslant L$ provided $\mu k^2 x_0 \ll 1$. The conditions of case (iii) can be reinterpreted as follows. First, $\mu k \ll 1$ can be rewritten $\mu k^2 \ll k$ so that the requirement is that the damping rate be much smaller than the wave frequency. Second, the condition $\mu k^2 L \gg 1$ requires $kL \gg 1$, so that the number of wavelengths in the computational domain is large. The latter requirement together with the condition $x_0 \ll L$ shows that constant damping can be used to achieve radiation boundary conditions but at a very high computational cost.

A more careful analysis of the maximum error than provided by (3.10) shows that there are two essential kinds of error in $v(x)$, which may be termed phase error and damping error. Phase error is that error induced on the frequency of the outgoing wave by the presence of the damping, while damping error is due to both damping of the outgoing wave and the unwanted reflected wave. In the limit (iii), phase error is negligible. In order to make the error of order $\varepsilon$ in the region $0 \leqslant x \leqslant x_0$, it is necessary that

$$e^{-\mu k^2 x_0/2} \doteq 1 - \varepsilon$$

so that the outgoing wave is only slightly damped, and that

$$e^{-\mu k^2 L} \doteq \varepsilon$$

so the unwanted reflected wave is greatly damped. In other words, we require

$$\frac{x_0}{L} \doteq \frac{2\varepsilon}{\ln(1/\varepsilon)}, \qquad \mu k^2 x_0 \sim 2\varepsilon.$$

With $\varepsilon = 0.01$, $k = x_0 = 1$, it follows that we must choose $L \sim 250$, $\mu \sim 0.02$. Thus, 99.6 % of the computational domain is wasted outside the region $0 \leqslant x \leqslant x_0$ of accuracy, hardly a satisfactory situation. This problem is even more severe in three space dimensions.

Another difficulty with simple application of damping concerns its effect on time-dependent problems. The long-time behavior of the general solution to (3.4)–(3.6) with $\mu(x) \equiv \mu$ is dominated by the right-most poles of the Laplace transform $\hat{v}(x, p)$ of $v(x, t)$ in the complex-$p$ plane. The boundary contribution to $\hat{v}(x, p)$ is of the form

$$\hat{v}_B(x, p) = \frac{k}{p^2 + k^2} \frac{\sinh p(L - x)/\sqrt{1 + p\mu}}{\sinh pL/\sqrt{1 + p\mu}}. \tag{3.11}$$

The poles of $\hat{v}_B(x, p)$ at $p = \pm ik$ give the persistent part of the solution as $t \to \infty$, namely, (3.6). However, $\hat{v}_B(x, p)$ [as well as $\hat{v}_I(x, p)$, that part of $\hat{v}(x, p)$ due to initial conditions] has other poles at

$$\frac{pL}{\sqrt{1 + p\mu}} = in\pi \qquad (n \neq 0) \tag{3.12}$$

or

$$p = -(\mu n^2\pi^2 \pm \sqrt{\mu n^4\pi^4 - 4L^2n^2\pi^2})/2L^2$$
$$\sim -\frac{\mu n^2\pi^2}{2L^2} \pm in\pi(\mu n^2L^{-2} \ll 1). \tag{3.13}$$

These transient modes decay at the rate $\mu n^2\pi^2/2L^2$. In particular, if $L = \pi$ the most persistent transient persists for the time $2/\mu$, which may be very long. These persistent transients may give errors much larger and persistent than those predicted by analysis of the steady state oscillation (see Fig. 1).

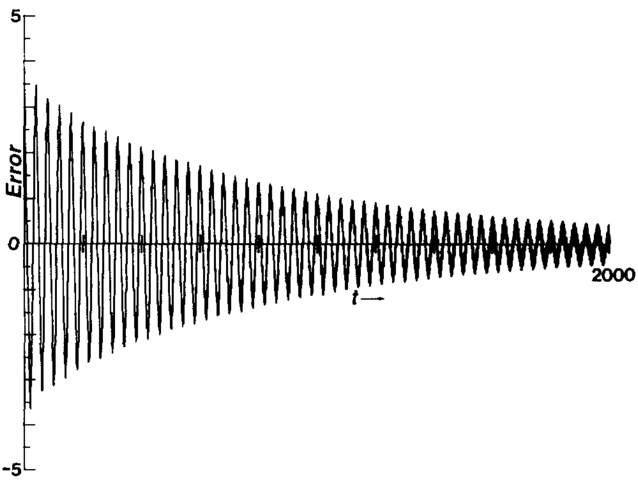


FIG. 1. A plot of the error $v(x, t) - \sin k(t - x)$ at $x = 1$ in the solution of (3.4)–(3.6) with $\mu = 0.1$, $\nu = 0$, $k = 2$, $L = 20$ and $0 \leqslant t \leqslant 2000$. Here $v(x, 0) = \frac{1}{5}x(L - x)$, $v_t(x, 0) = 0$. Observe the long decay time of the transient error to its time asymptotic value 0.18 given by (3.9). This behavior is due to the mode (3.13) with $n = 1$.

## 4. Sponge Layers

Allowing the damping coefficients $\mu(x)$ and $\nu(x)$ in (3.4) to vary with $x$ may give significant improvements in the results obtained by damping. For example, Aarakawa and Mintz [11] considered application of damping in a sponge layer $x_0 < x < L$ by choosing $\nu(x) = 0$ and

$$\begin{aligned} \mu(x) &= 0, \qquad 0 \leqslant x \leqslant x_0, \\ &= \mu, \qquad x_0 < x \leqslant L, \end{aligned} \tag{4.1}$$

in order to minimize wave reflections in a general circulation model of the atmosphere.

In the spatial region $0 \leqslant x \leqslant x_0$, where the damping coefficients $\mu(x)$ and $\nu(x)$ both vanish, the steady oscillatory solution to (2.1) with $L^2 = -k^2$ or (3.4) is, aside from a constant phase and amplitude shift,

$$v(x, t) = \operatorname{Im}\left[\frac{1}{1+R} e^{ik(t-x)} + \frac{R}{1+R} e^{ik(t+x)}\right], \qquad 0 \leqslant x \leqslant x_0, \tag{4.2}$$

where $R$ is the reflection coefficient. Note that $|R|$ measures the amplitude of the reflected wave. With damping of the form (4.1) and the boundary condition $v(L, t) = 0$ applied, the solution to (3.4) that is continuous at $x_0$ gives the reflection coefficient

$$R = -e^{-2ikx_0} \frac{\alpha - ik \tan \alpha(L - x_0)}{\alpha + ik \tan \alpha(L - x_0)}, \tag{4.3}$$

where $\alpha = k/\sqrt{1 + i\mu k}$.

Some limiting cases of (4.3) are of interest:

(i) If $\mu k \to \infty$ then $|R| \to 1$. The abrupt change of damping coefficient across the boundary of the sponge layer reflects waves as if the boundary condition $v(x_0, t) = 0$ were applied.

(ii) For any fixed $k$ and sponge layer with $x_0 < L$, it is possible to find discrete complex values of $\mu$ such that $R = 0$ so there are no reflected waves. However, as shown by Arakawa and Mintz [11], slight detuning of $k$ from these discrete resonances gives large reflections. This sensitivity to the wave parameters is unacceptable.

(iii) If $\mu k \ll 1$ then

$$|R| \sim e^{-\mu k^2 (L - x_0)} \tag{4.4}$$

with a correction of order $\frac{1}{4}\mu k$. Thus, if it is also true that $\mu k^2 (L - x_0) \gg 1$, small reflection results. Under these conditions, the propagation domain $0 \leqslant x \leqslant x_0$ can be chosen arbtrarily long without further wave reflections.

In order to achieve 1% wave reflections by this method, we choose $\frac{1}{4}\mu k \lesssim 10^{-2}$ and $\mu k^2(L - x_0) \gtrsim \ln 100 = 4.6$. Thus, we require $k(L - x_0) \gtrsim 100$, so that the sponge layer $x_0 \leqslant x \leqslant L$ should contain at least 15 full wavelengths. This restriction severely limits the utility of the simple sponge layer (4.1), especially for multi-dimensional problems. (The methods introduced later give 1% wave reflections with less than one wavelength contained within the sponge layer.)

The damping provided by $\nu(x)$ (Newtonian cooling) has an advantage over that provided by $\mu(x)$ (viscous damping). For a given choice of damping coefficients, a broader spectrum of waves can be damped by Newtonian cooling than by viscous damping. With $\nu(x) \equiv \nu$ and $\mu(x) \equiv \mu$, there are steady wave solutions of (2.4) of the form

$$v(x) = e^{ikt - i\alpha x},$$

where

$$\alpha = k\left(\frac{1 - i\nu/k}{1 + i\mu k}\right)^{1/2}. \tag{4.5}$$

The damping rate is $-\text{Im}\,\alpha$. With $\nu = 0$,

$$\begin{aligned} -\text{Im}\,\alpha &\sim \tfrac{1}{2}\mu k^2 \qquad (k \to 0),\\ -\text{Im}\,\alpha &\sim (k/2\mu)^{1/2} \qquad (k \to \infty) \end{aligned} \tag{4.6}$$

while, with $\mu = 0$,

$$\begin{aligned} -\text{Im}\,\alpha &\sim (\tfrac{1}{2}\nu k)^{1/2} \qquad (k \to 0),\\ -\text{Im}\,\alpha &\sim \tfrac{1}{2}\nu \qquad (k \to \infty). \end{aligned} \tag{4.7}$$

For fixed $\mu$ or $\nu$, $\text{Im}\,\alpha$ has less variation as a function of $k$ when Newtonian cooling is used. Also, as $\nu \to \infty$ with $\mu = 0$ for fixed $k$, $-\text{Im}\,\alpha \to \infty$ so large damping can be achieved, while if $\nu = 0$ and $\mu \to \infty$, $-\text{Im}\,\alpha \to 0$, so only weak damping results.

An improvement in the sponge layer idea is obtained by allowing the damping coefficients to be functions of $x$ [12]. Suppose that $\mu(x) = 0$ and $\nu(x)$ is of the form

$$\begin{aligned} \nu(x) &= 0, \qquad 0 \leqslant x \leqslant x_0,\\ &= \nu_0(x), \qquad x_0 < x \leqslant L, \end{aligned} \tag{4.8}$$

where $\nu_0(x)$ is a function of $x$. If $\nu_0(x_0) = 0$ then it is possible to avoid wave reflections at the interface $x = x_0$. If $\nu_0(x)$ does not vary too rapidly with $x$, only small total reflections result.

Result (4.2) for the wave amplitude in the region $0 \leqslant x \leqslant x_0$ is still valid for the damping coefficient (4.8) even though the reflection coefficient $R$ is no longer given by (4.3). In general, $|R|$ is determined by the formula

$$|R| = \left|\frac{ikv + v_x}{ikv - v_x}\right|_{x = x_0}. \tag{4.9}$$

In Fig. 2, we plot the reflection coefficient $|R|$ for damping functions of the form

$$v_0(x) = A(n+1)(x-x_0)^n/(L-x_0)^n \tag{4.10}$$

as a function of $A$ for $n = 1, 2, 3, 4$. Here the boundary condition $v(L, t) = 0$ is applied and $k(L - x_0) = 3\pi$ so that there are $1\frac{1}{2}$ wavelengths within the sponge layer. When $A = 0$, perfect reflection, $|R| = 1$, results. With increasing damping coefficient $A$, the error $|R|$ due to waves reflected from the boundary decreases. Note that the form of $v_0(x)$ is chosen so that $\int_{x_0}^{L} v_0(x)\,dx = A$ (see the WKB analysis below).

In Figs. 3 and 4 similar plots are given for the cases $k(L - x_0) = 2\pi$ and $k(L - x_0) = \pi$, respectively, so there are 1 and $\frac{1}{2}$ wavelengths, respectively, lying within the sponge layer. Observe that the reflection coefficients are not necessarily monotonically decreasing with increasing $A$, because reflections from within the sponge layer become important if too much damping is applied. It is apparent that there is a range of damping coefficients $A$ and exponents $n$ that gives wave reflections of no more than a few percent so long as the sponge layer is at least one wavelength long.

An asymptotic analysis of the effectiveness of sponge layers can be given using WKB theory [13]. For a steady oscillation of the form

$$v(x, t) = \mathrm{Im}[v(x)e^{ikt}],$$

(3.4) reduces to

$$v_{xx} + k^2[1 - iv(x)/k]v = 0. \tag{4.11}$$

The WKB solution to (4.11) satisfying $v(L, t) = 0$ is

$$v(x, t) \sim a[1 - iv(x)/k]^{-1/4} \sin\left[k \int_x^L \sqrt{1 - iv(t)/k}\,dt\right]. \tag{4.12}$$

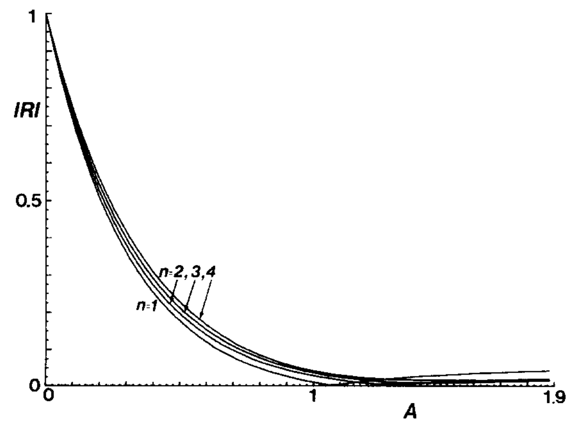


FIG. 2. A plot of the reflection coefficient $|R|$ for the damping function (4.10) with $n = 1, 2, 3, 4$ as a function of $A$. Here the reflecting boundary condition $v(L, t) = 0$ is applied and $k(L - x_0) = 3\pi$ so there are $1\frac{1}{2}$ wavelengths in the damping region.

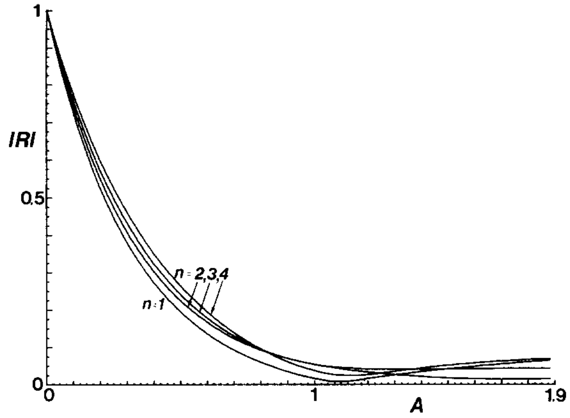


FIG. 3. Same as Fig. 2 except that $k(L-x_0)=2\pi$ so there is 1 wavelength in the damping region.

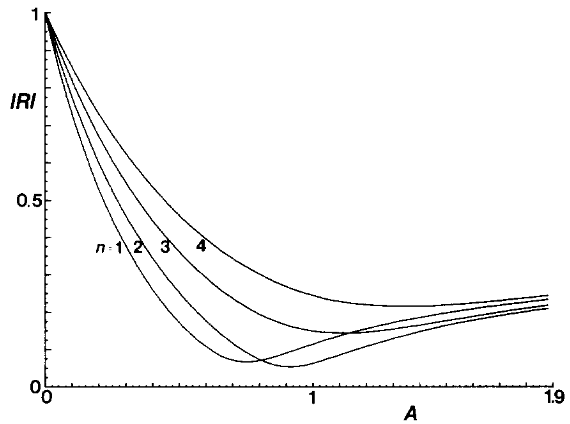


FIG. 4. Same as Fig. 2 except that $k(L-x_0)=\pi$ so there is $\frac{1}{2}$ wavelength in the damping region.

Identifying terms in (4.2) gives

$$|R| = \exp\left(2k \operatorname{Im} \int_{x_0}^{L} \sqrt{1 - i\nu(t)/k}\, dt\right). \tag{4.13}$$

For fixed damping function $\nu(x)$, the WKB result (4.13) is asymptotically exact in the limit $k \to \infty$; in fact, the condition for validity of the WKB result (4.12) is that

$$\nu' \ll k^2(1 + \nu/k)^{3/2} \tag{4.14}$$

uniformly on the interval $x_0 \leqslant x \leqslant L$.

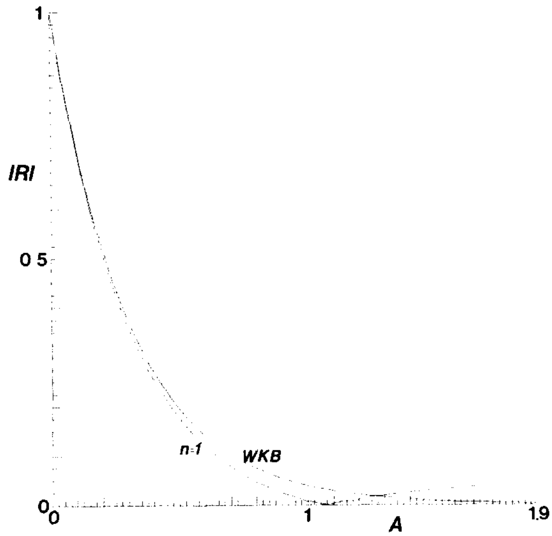


FIG. 5. Comparison between the WKB result (4.13) and the exact reflection coefficient $R$ as a function of $A$ in (4.10). Here the damping function (4.10) is used with $n = 1$, $k(L - x_0) = 3\pi$ so there are $1\frac{1}{2}$ wavelengths in the damping region.

In Fig. 5, we compare the WKB prediction (4.13) with the exact reflection coefficient $|R|$ for the damping function (4.10) with $n = 1$, $k(L - x_0) = 3\pi$ as a function of the coefficient $A$ in (4.10). For small $A$, the agreement is very good, but the WKB result does not exhibit the increase of $|R|$ with increasing $A$ at large $A$ due to wave reflections from the sponge. As $n$ increases in (4.10), the WKB approximation works better.

Observe that as $k \to \infty$ in (4.13)

$$|R| \sim \exp\left(-\int_{x_0}^{L} v(t)\, dt\right) = e^{-A}, \tag{4.15}$$

explaining the weak dependence of $|R|$ on $n$ observed in Figs. 2, 3.

## 5. Sponge Layers with Absorbing Boundary Conditions

While it is possible to adjust the variable damping in sponge layers to minimize wave reflections for steady oscillations, the problem with transient behavior pointed out at the end of Section 3 remains. When reflecting boundary conditions like $v(L, t) = 0$ are applied, transient energy cannot be radiated from the domain but can only be dissipated by damping. With weak damping and reflecting boundary conditions, transients persist for long times.

A substantial improvement in both transient and steady state behavior can be obtained by combining damping with absorbing boundary conditions. Damping reduces the amplitude of the outgoing waves as well as the amplitude of those waves that are not absorbed at the boundary. For example, if we repeat the Laplace transform analysis whose result was (3.11) for the case in which the reflecting condition $v(L, t) = 0$ is replaced by the *approximate* absorbing boundary condition

$$\delta v_t + v_x = 0 \tag{5.1}$$

applied at $x = L$, the Laplace transform solution becomes

$$\hat{v}_B(x, p) = \frac{k\left[\delta\sqrt{1+p\mu}\sinh\frac{p}{\sqrt{1+p\mu}}(L-x) + \cosh\frac{p}{\sqrt{1+p\mu}}(L-x)\right]}{(p^2+k^2)\left[\delta\sqrt{1+p\mu}\sinh\frac{pL}{\sqrt{1+p\mu}} + \cosh\frac{pL}{\sqrt{1+p\mu}}\right]}. \tag{5.2}$$

The poles of $\hat{v}_B(x, p)$ are at $p = \pm ik$ and

$$\coth\frac{pL}{\sqrt{1+p\mu}} = -\delta\sqrt{1+p\mu}. \tag{5.3}$$

As $\mu \to 0+$, the slowest decaying transient modes behave like

$$p \sim \frac{1}{2L}\ln\frac{1-\delta}{1+\delta} \qquad (0 < \alpha < 1) \tag{5.4}$$

so that transients decay within a finite time in the limit $\mu \to 0+$ so long as $\delta > 0$.

The simultaneous application of absorbing boundary conditions and damping should be done very carefully because of possible interference between them. The proper absorbing boundary condition should account for the change in propagation characteristics induced by the damping. For the model problem (3.4) with $\nu(x) \equiv \nu$ and $\mu(x) \equiv 0$, the absorbing boundary condition is not $u_x + iku = 0$ but rather

$$u_x + i\sqrt{k^2 - ik\nu}\,u = 0$$

or, as $\nu \to 0+$,

$$u_x + iku + \tfrac{1}{2}\nu u = 0. \tag{5.5}$$

Another way to minimize interference between absorbing boundary conditions and the sponge layers is to choose damping coefficients $\nu(x)$ that approach zero at the artificial boundary $x = L$. A further improvement on this idea, with numerical results, is presented in Section 6.

## 6. Sponge Filters

The sponge layer idea discussed in Sections 4 and 5 can be improved by adjusting the damping to the character of the waves being modelled. A hint of this idea was given in Section 5, where we showed that the combination of absorbing boundary conditions with sponge layers could remove some undesirable transients. Several further improvements are possible.

First, we consider modifications of damping that selectively filter only the undesirable wave components. For example, the modified one-dimensional wave equation

$$u_{tt} = u_{xx} - \nu(x)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u \tag{6.1}$$

selectively filters the left-going waves by the Newtonion cooling coefficient $\nu(x) > 0$, while right-going waves propagate undamped. With this kind of "sponge filter," a relatively large damping effect on waves incoming from $x = +\infty$ can be achieved without distorting or reflecting the outgoing waves.

For more complicated problems, similar modifications of the damping terms can be made. For problem (2.1), the analog of the damping operator

$$\nu(x)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u$$

in (6.1) is

$$\nu(x)\left(\frac{\partial}{\partial x} + L\right) u, \tag{6.2}$$

where $L$ is a square root of $L^2$ in the sense of pseudodifferential operators. Instead of $L$ in (6.2), we may use any convenient low-order numerical approximation to $L$. This approach applies to a large class of dispersive wave problems. First, we isolate the incoming and outgoing waves. Then we determine forms of the damping terms that affect only incoming waves while letting outgoing waves leave the domain unscattered.

To test these ideas, let us consider the Fourier transformed two-dimensional wave equation (2.1) with (2.4):

$$\left(\frac{\partial^2}{\partial x^2} + m^2\right) u = 0, \tag{6.3}$$

where $m^2 = \omega^2 - l^2$. A suitable model for study of sponge filters to impose radiation boundary conditions as $x \to +\infty$ is

$$\left(\frac{\partial^2}{\partial x^2} + m^2\right) u - \nu(x)\left(\frac{\partial}{\partial x} + i\bar{m}\right) u = 0, \tag{6.4}$$

where $\nu(x)$ is a suitable damping function and $\bar{m}$ is an approximation to $\sqrt{\omega^2 - l^2}$. [In more difficult problems, only an approximation $\bar{m}$ to $m$ will be available because $m^2$ is a differential operator so $m$ is a pseudodifferential operator as in Section 2.] When $\bar{m} \simeq m$, the outgoing wave is almost unaffected by the damping while the incoming wave can be highly damped. If $l \ll \omega$ then we expect that good approximations $\bar{m}$ to $m$ can be routinely obtained but if $l \simeq \omega$ so $m \ll \omega$, then approximation of $m$ is more difficult. Numerical results are presented later in this section and in Section 7.

Another way to improve the absorption of outgoing waves is to modify the propagation characteristics of the waves near the boundary. If we have determined an approximate absorbing boundary condition

$$\left(\frac{\partial}{\partial x} + i\bar{m}\right) u = 0 \tag{6.5}$$

for (6.3) then reflections will be decreased if we smoothly modify (6.3) as $x$ approaches the boundary into the equation

$$\left(\frac{\partial^2}{\partial x^2} + \bar{m}^2\right) u = 0, \tag{6.6}$$

for which (6.5) is an *exact* absorbing boundary condition.

Let us now test all these ideas for imposition of radiation boundary conditions. There are three methods we want to test, namely:

(i) absorbing boundary conditions (Section 2),

(ii) sponge filters like (6.4),

(iii) propagation modifications like (6.6).

A good model for these tests is the following generalization of (6.4):

$$\left\{\frac{\partial^2}{\partial x^2} + \kappa(x)\bar{m}^2 + [1 - \kappa(x)]m^2 - \nu(x)\left[im + \eta\frac{\partial}{\partial x}\right]\right\} u = 0 \tag{6.7}$$

with the boundary condition

$$\left[\frac{\partial}{\partial x} + i\bar{m}\right] u = 0 \qquad \text{at} \quad x = L. \tag{6.8}$$

The reflection coefficient at $x = x_0$ is given by (4.9):

$$|R| = \left|\frac{imu + u_x}{imu - u_x}\right|_{x=x_0}. \tag{6.9}$$

We choose $m = \beta\bar{m}$, so $\beta \neq 1$ represnts the error in the approximation $\bar{m}$. The parameter $\eta$ is used to simulate how well we can tune the sponge filter in (6.7) to dissipate only outgoing waves. Usually, it is reasonable to choose $\beta = \eta$ because the same approximation to the dispersion relation for outgoing waves can be used for both the sponge filter in (6.7) and the absorbing boundary condition in (6.8). The function $\kappa(x)$ is used to impose smoothly the propagation modification (6.6) [when $\kappa(x) = 1$] to the exact equation (6.3) [which holds when $\kappa(x) = 0$]. We choose $\kappa(x) = 0$, $0 \leqslant x \leqslant x_0$, and

$$\kappa(x) = B\left(\frac{x - x_0}{L - x_0}\right)^n \qquad (x_0 \leqslant x \leqslant L) \tag{6.10}$$

for various $n$, so $\kappa(x_0) = 0$ and $\kappa(L) = B$, so $B = 1$ represents full adjustment of the propagation speed to the absorbing boundary condition. We also choose the Newtonian cooling function $\nu(x)$ to be

$$\begin{aligned} \nu(x) &= 0 && (0 \leqslant x \leqslant x_0) \\ &= A(x - x_0)^n (L - x)(L - x_0)^{-n-2}(n+1)(n+2) && (x_0 \leqslant x \leqslant L) \end{aligned} \tag{6.11}$$

so $\nu(x_0) = \nu(L) = 0$ and $\int_{x_0}^{L} \nu(x)\, dx = A$.

In Fig. 6, we plot the reflection coefficient $|R|$ versus $A$ in (6.11) for $m(L - x_0) = 2\pi$, $B = 0$, $\beta = 0.5$, $\eta = 0$ so there is one wavelength in the damping region, no propagation modification, a sponge layer (but not a sponge filter) and $m = \frac{1}{2}\bar{m}$ in the absorbing boundary codition. In Fig. 7, a similar plot is made with the only change being $\eta = 0.5$ so a sponge filter is used. In Fig. 8, a similar plot with $B = 1$ and $\eta = 0.5$ is given so both propagation modification and a sponge filter are used.

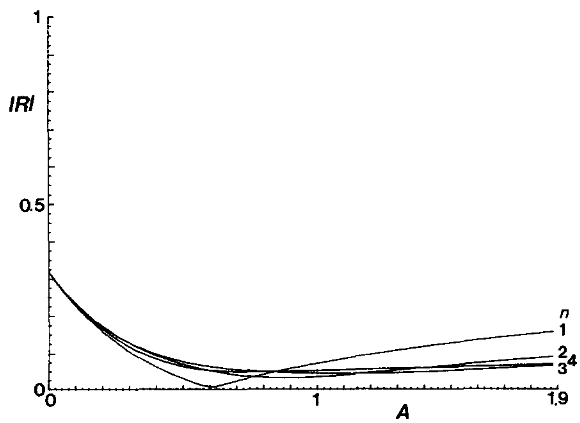


FIG. 6. A plot of the reflection $|R|$ vs $A$ for the model problem (6.7)–(6.8) with $B = 0$ in (6.10), $m = \frac{1}{2}\bar{m}$, $\eta = 0$ and $\nu(x)$ given by (6.11) with $n = 1, 2, 3, 4$. Here $m(L - x_0) = 2\pi$ so there is 1 wavelength within the damping region.

It is apparent from Figs. 6–8 that by combination of all the devices described here we can achieve broadband radiation characteristics that minimize wave reflections. Comparison of the results plotted in Figs. 7 and 8 suggest that either a sponge filter or propagation modification should be used (in addition to absorbing boundary conditions), but there is not much to be gained by their simultaneous application.

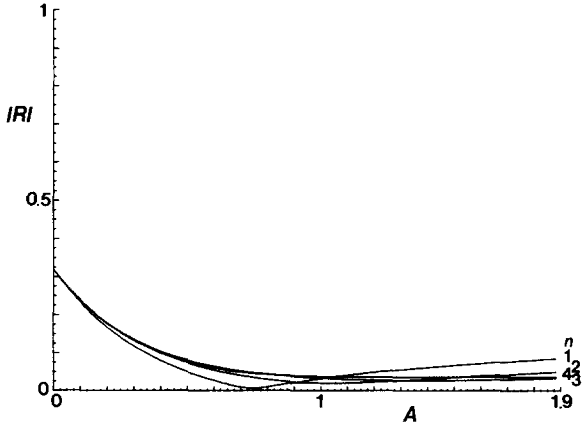


FIG. 7. Same as Fig. 6 except $\eta = 0.5$ so a sponge filter is applied. For moderate to large $A$, the reflection coefficients are decreased by about a factor 2 relative to those obtained without the sponge filter.

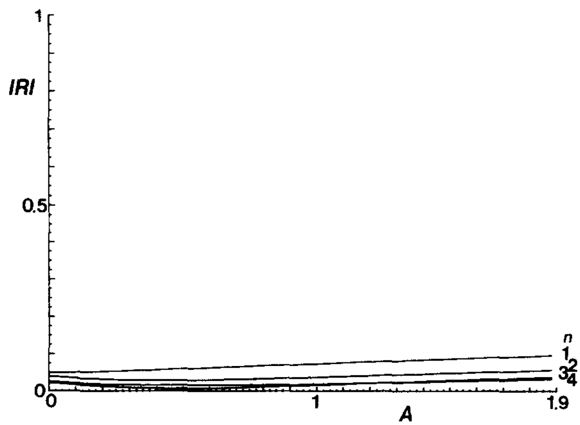


FIG. 8. Same as Fig. 6 except $\eta = 0.5$ and $B = 1$ so the propagation modification (6.10) is applied.

## 7. Application to the Klein–Gordon Equation

Here we test the ideas discussed above on the solution of the Klein–Gordon equation

$$u_{tt} = u_{xx} - k^2 m^2 u, \tag{7.1}$$

$$u(0, t) = \sin kt \tag{7.2}$$

with radiation boundary conditions at $x = \infty$. Aside from its intrinsic interest, this problem is also a model of the multi-dimensional wave equation (with transverse wavenumber $km$). As $m \to 1-$, the $x$-wavenumber decreases and the wave propagation direction becomes increasingly oblique. For $m > 1$, the waves are evanescent. Absorbing boundary conditions will not work well in the latter cases.

We solve (7.1)–(7.2) using a sponge filter together with absorbing boundary conditions at $x = X$:

$$w_{tt} = w_{xx} - k^2 m^2 w - \nu(x)(w_t + w_x) = 0, \tag{7.3}$$

$$w_t + w_x = 0 \qquad \text{at} \quad x = X \tag{7.4}$$

with $\nu(x)$ given by (6.11) with $L = X$, $x_0 = X - \pi$, $n = 2$ [so damping is only applied within an interval of length $\pi$]. The initial conditions are $w = w_t = 0$ for $0 \leqslant x \leqslant X$ and the boundary condition $w(0, t) = \sin kt$ is applied for all $t > 0$.

In Figs. 9 and 10, we plot the envelope of the error $|w - u|$ at $x = X - \pi$ obtained by numerical solution of (7.3)–(7.4) as a function of time $t$. The envelope removes the rapid oscillations of the error with frequency $k$. The error vanishes for $t < X - \pi$

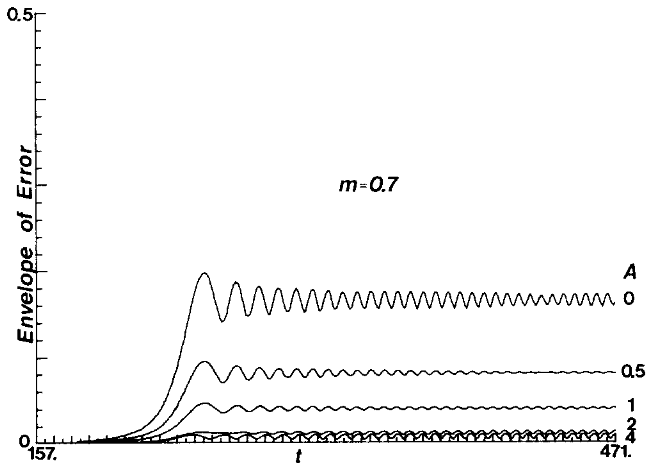


Fig. 9. Envelope of errors $|w(x, t) - u(x, t)|$ at $x = X - x$ vs $t$. Here $u(x, t)$ is the solution of the Klein–Gordon equation (7.1) with radiation boundary conditions and $w(x, t)$ is the solution of (7.3)–(7.4). Here $m = 0.7$, $k = 4$, $X = 50\pi$ and $\nu(x)$ is given by (6.11).

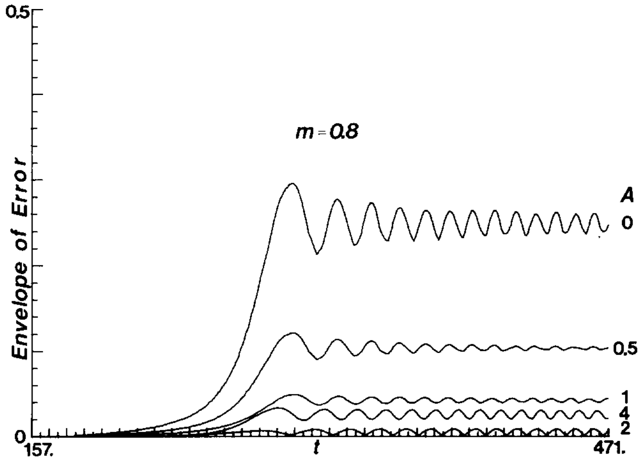


FIG. 10. Same as Fig. 9 except $m = 0.8$.

because the wave precursor travelling with speed 1 arrives at $x = X - \pi$ when $t = X - \pi$. In Fig. 9, the error is plotted when $m = 0.7$, $k = 4$, and $X = 50\pi$ for various values of $A$. In this case the $x$-wavelength $2\pi/k\sqrt{1 - m^2} = 2.20$ so there are about 1.4 wavelengths within the damping region $X - \pi < x < X$. Observe that as the damping coefficient increases from no damping to $A = 4$ the error decreases from 20% to less than 1%. Finally, in Fig. 10 a similar plot is given for $m = 0.8$ and $k = 4$, $X = 50\pi$. In this case, there are about 1.2 wavelenghts in the damping region. Notice that significant errors do not arrive with the precursor but travel with a speed of roughly $\sqrt{1 - m^2}$ in the $x$-direction.

## ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant ATM-7817092 and the Office of Naval Research under Contract N00014-77-C-0138.

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