MODELING EQUATORIAL OCEAN CIRCULATION

Julian P. McCreary, Jr.

Nova University Oceanographic Center, Dania, Florida 33004

1. INTRODUCTION

The field of equatorial oceanography has grown tremendously in the past decade. One reason for this growth has been the discovery of a rich variety of phenomena in the region. Another is the realization that sea-surface temperature (SST) anomalies in the equatorial oceans have a significant effect on global climate. Not surprisingly, a large number of models have been developed to study equatorial dynamics.

1.1 Observations

Since the focus of this paper is on the modeling of equatorial circulation, observational studies are not reviewed extensively here. Rather, this subsection briefly surveys some observations that have provided the stimulus for the development of equatorial theory. Two recent reviews discuss observations in greater detail. McCreary et al. (1981) is a collection of short articles on the subject, and Knox & Anderson (1984) is a comprehensive review of phenomena from all three tropical oceans.

EQUATORIAL CURRENTS  The Equatorial Undercurrent is a geostrophically balanced, subsurface, eastward jet that lies in the pycnocline just beneath the surface mixed layer. Since its rediscovery in the early 1950s, it has been detected on almost every cruise to the equatorial Atlantic and Pacific Oceans; the sole exception occurred during the 1982/83 El Niño, when the flow in the upper ocean in the central Pacific was westward for a period of about one month (Firing et al. 1983). Philander (1973) reviews early observations of the Undercurrent, while Firing et al. (1981) report a recent set of observations of the Pacific Equatorial Undercurrent taken for a period of time lasting longer than one year.

On several occasions an ageostrophic equatorial undercurrent, distinct
from the deeper one, has been observed in the thick surface layer of the western Pacific (Hisard et al. 1970, Donguy et al. 1984). On these occasions the surface-layer flow is not at all depth-independent; in the upper part of the layer there is westward drift driven by the prevailing trade winds, whereas in the lower part the flow is eastward. These currents respond rapidly to changes in the wind. For example, Hisard et al. reported that in April 1967 the trade winds suddenly reversed for a period of time. After eight days of westerly winds the sense of the surface-layer velocity shear was completely reversed, with eastward flow at the surface and westward flow at depth. The deeper, geostrophically balanced undercurrent was not significantly affected by the wind change, indicating that the dynamics of the two undercurrents are considerably different.

Two other equatorial currents are the subsurface countercurrents. These currents are geostrophically balanced, eastward flows located nearly symmetrically about the equator at depths somewhat greater than that of the Equatorial Undercurrent. At some locations they are entirely separate from the Undercurrent, but at others they are contiguous with it. They are appreciable flows, attaining speeds and volume transports that are typically 20 cm s\(^{-1}\) and 5–10 sverdrups, respectively. Tsuchiya (1972, 1975) noted their existence in the Pacific Ocean, and Cochrane et al. (1979) found them in the Atlantic Ocean as well. McPhaden (1984) briefly reviews these papers and discusses other observations of subsurface countercurrents.

**EQUATORIALLY TRAPPED AND UNSTABLE WAVES** Variability in the equatorial oceans is often interpreted as being due to linear, equatorially trapped waves. For example, Wunsch & Gill (1976) pointed out that sea-level variability at periods of about 10 days appears to be caused by resonant, equatorial gravity waves. Duing & Hallock (1979) and Hallock (1979) suggested that meanders of the Atlantic Equatorial Undercurrent with a period of about 16 days are wind-generated, Rossby-gravity waves. Knox & Halpern (1982) reported a pulse of currents and sea level that propagated eastward across the Pacific Ocean with a speed approaching 3 m s\(^{-1}\), and they suggested that the event was an equatorial Kelvin wave.

Luyten & Swallow (1976) first reported the existence of deep currents throughout the water column in the western Indian Ocean. Similar deep currents have since been observed in the Pacific and Atlantic Oceans as well. These currents are characterized by a broad spectrum of vertical wavelengths, have time scales of one month and longer, and are trapped within a few degrees of the equator. Linear, equatorial wave theory has been used with some success to interpret various aspects of these observations (see, for example, Eriksen 1981). Particularly interesting is the presence of vertically propagating signals in the data. Weisberg & Horigan (1981) in the
Atlantic Ocean and O'Neill (1982) in the Indian Ocean interpreted variability at periods of 1–2 months to be due to vertically propagating Rossby-gravity waves. Luyten & Roemmich (1982) noted upward phase propagation of current variability at the semiannual period in the western Indian Ocean and suggested that this signal was likely to be a Rossby wave. Lukas & Firing (1984) presented evidence for a vertically propagating Rossby wave at the annual period in the Pacific Ocean.

Legeckis (1977) and Legeckis et al. (1983) reported the presence of cusplike waves located a few degrees north of the equator in the eastern and central Pacific Ocean. The waves were visibly apparent in satellite images of SST. They propagated westward and had a wavelength of about 1000 km and a period of about one month. There is no evidence of a spectral peak in the wind field with a similar wave number and frequency. It is likely, then, that this signal is an unstable wave, as suggested by Philander (1978b).

**REMOTE FORCING**

A remarkable property of the equatorial oceans is that wind fluctuations in one part of the basin can quickly affect the state of the ocean at locations thousands of kilometers away. A well-known example of a remotely forced event is the appearance of warm SST anomalies in the eastern Pacific during El Niño. Another example is the annual cycle of ocean circulation in the eastern Atlantic.

Bjerknes (1966, 1969) first noted that remote winds at Canton Island were highly correlated with SST off Peru at the time scales associated with El Niño. It is now clear that El Niño is one aspect of changes in ocean circulation throughout the Pacific Ocean and of changes in the atmosphere on a global scale. Rasmussen & Carpenter (1982) and Wyrtki (1977, 1979, 1984) document the changes in SST, sea level, and wind stress that occur in the tropical Pacific during El Niño events, while Horel & Wallace (1981) summarize the global changes that occur in the atmosphere. The collection of papers in Witte (1983) discusses events that took place during the 1982/83 El Niño.

During the Northern Hemisphere summer, SST in the eastern Atlantic cools by about 5°C along the equator and the west coast of Africa. This cooling is particularly interesting because the annual cycle of the local winds is weak in the region. Moore et al. (1978) first suggested that the annual cooling might be remotely forced by the strong annual cycle of the equatorial zonal wind field in the western Atlantic. In support of this idea, Servain et al. (1982) demonstrated that SST in the eastern Atlantic is highly correlated with winds from the western Atlantic, but it is much less correlated with the local winds. Picaut (1983) discussed evidence that the upwelling signal along the coast of Africa is a remotely forced, propagating wave.
1.2 Theories

This paper reviews the models that have been used to study phenomena like those discussed above. The purpose of the paper is twofold: to present the basic principles of equatorial dynamics as succinctly as possible, and to survey recent developments in the field. It is hoped that those readers who are not familiar with the subject will find this paper to be a useful introduction, and that readers who are already familiar with the topic will find it to be a convenient reference source. Section 2 organizes the various models according to dynamical sophistication. Sections 3–5 discuss unforced solutions, solutions forced by switched-on winds, and solutions forced by periodic winds, respectively. Finally, Section 6 outlines the directions of current research in the study of El Niño, a subject of considerable interest at the present time. As we shall see, linear models are extremely useful for identifying important processes involved in equatorial dynamics. For this reason, some linear solutions are written down explicitly and discussed at length. Linear solutions are compared with nonlinear ones whenever possible.

A number of other review articles of equatorial models already exist to supplement this one. O'Brien (1979), Leetmaa et al. (1981), Cane & Sarachik (1983), and Knox & Anderson (1984) all present useful introductions to the subject. McCreary (1980) discusses the mathematical techniques used for finding analytic solutions to a variety of equatorial problems, and O'Brien et al. (1981) review models of El Niño. Gill (1983a) devotes a chapter of his book to the equatorial dynamics of the atmosphere and the ocean. Finally, the collection of articles in Nihoul (1985) is a good introduction to the new field of coupled ocean-atmosphere models.

2. EQUATIONS OF MOTION

A wide variety of ocean models have been used to study equatorial circulation. They range in dynamical complexity from simple, linear, surface-layer models to sophisticated, nonlinear, continuously stratified models. The sophisticated models are remarkably successful at producing realistic solutions. The simple models have proven to be invaluable tools for isolating important physical processes at the equator, and they have also produced quite realistic solutions. It is fair to say that our understanding of equatorial dynamics has progressed through a careful comparison of the solutions to models of all types. This section describes commonly used models and organizes them, as much as is possible, according to increasing sophistication.
2.1 Surface-Layer Models

The density structure of the equatorial ocean often consists of a well-mixed surface layer of constant density above a sharp pycnocline. Such a structure is almost always present in the western Pacific and Atlantic Oceans, where the mixed layer is 100–150 m deep. A number of equatorial models consider the ocean circulation in just this surface layer alone. Their underlying assumption is that the surface layer is decoupled from the deep ocean by the sharp pycnocline.

**CONSTANT-THICKNESS MODELS** Some of the earliest equatorial models assume the surface layer has a constant thickness $H$. A simple set of equations describing the flow in such a layer is the linear set

$$\begin{align*}
    u_t - f v + p_x &= (v u_z)_z + v_h \nabla^2 u, \\
    v_t + f u + p_y &= (v v_z)_z + v_h \nabla^2 v, \\
    u_x + v_y + w_z &= 0, \quad p_z = 0,
\end{align*}$$

where $u$, $v$, and $w$ are zonal, meridional, and vertical velocities, respectively; $p$ is the pressure; $f$ is the Coriolis parameter; and $v_h$ and $v$ are coefficients of horizontal and vertical eddy viscosity, respectively. Boundary conditions at the ocean surface are

$$\begin{align*}
    v u_z &= \tau^x, \quad v v_z &= \tau^y, \quad w = 0 \quad \text{at} \quad z = 0, \\
\end{align*}$$

and at the bottom of the layer are

$$\begin{align*}
    u_z = v_z = w = 0 \quad \text{at} \quad z = -H, \\
\end{align*}$$

where $\tau^x$ and $\tau^y$ are zonal and meridional components of surface wind stress. The bottom conditions ensure that the deep ocean does not influence the flow in the surface layer in any way.

It is possible to solve for the depth-averaged flow $(\bar{u}, \bar{v})$ independently from the shear flow $(u' = u - \bar{u}, v' = v - \bar{v})$. Integrating (1) over the water column and dividing by $H$ gives

$$\begin{align*}
    \bar{u}_t - f \bar{v} + p_x &= \tau^x/H + v_h \nabla^2 \bar{u}, \\
    \bar{v}_t + f \bar{u} + p_y &= \tau^y/H + v_h \nabla^2 \bar{v}, \\
    \bar{u}_x + \bar{v}_y &= 0.
\end{align*}$$

The difference between Equations (4) and (1) gives the set that describes the shear flow,

$$\begin{align*}
    u'_t - f v' &= - \tau^y/H + (v u'_z)_z + v_h \nabla^2 u', \\
    v'_t + f u' &= - \tau^x/H + (v v'_z)_z + v_h \nabla^2 v', \\
    u'_x + v'_y + w_z &= 0,
\end{align*}$$
and Equations (5) are solved subject to boundary conditions (2) and (3) with \( u \) and \( v \) replaced by \( u' \) and \( v' \). Note that the two flow components are entirely separate from each other; that is, there are no terms in (4) and (5) that couple the components together. This convenient separation is possible only because the system neglects bottom stress and nonlinear terms.

The interesting and instructive property of this system is that the dynamics of each component are very different. Equations (4) are just the equations describing a barotropic ocean, and Equations (5) do not involve any pressure gradients. Each component responds very differently to the wind. For example, in response to a switched-on wind, the depth-averaged flow adjusts to a nonlocal Sverdrup balance by radiating barotropic waves, whereas the shear flow rapidly adjusts to a completely local balance (see the discussion in Section 4.1).

Various sets of equations similar to (1) have been studied. Stommel (1960) found steady-state solutions to (5) without horizontal mixing (see the discussion of Figure 2). Gill (1971) and McKee (1973) later included horizontal mixing and found solutions in a parameter range where horizontal mixing was a dominant process. Charney (1960) and Charney & Spiegel (1971) adopted no-slip boundary conditions (that is, \( u = v = 0 \) at \( z = -H \)), thereby allowing the deep ocean to have a strong influence (unrealistically strong) on the surface layer. Both of these latter studies also added nonlinear terms to Equations (1), but otherwise they simplified the system by reducing it to two dimensions in \( y \) and \( z \), retaining only the effects of \( p_x \) by specifying it externally to be \( \tau^x/H \).

These models are useful in spite of their dynamical simplicity. One reason is that they are able to simulate features of equatorial flows. As noted in the Introduction, very strong shear flows do exist in the constant-density surface mixed layer near the equator. These currents respond rapidly to the wind, and they often fluctuate independently from currents in the deeper ocean. Another reason is that, as we shall see, shear flows similar to (5) also appear as parts of more sophisticated reduced-gravity and continuously stratified models.

### Reducéd-Gravity Models

A prominent feature of the equatorial ocean is that the sharp, near-surface pycnocline can move vertically a considerable distance in a short period of time (of the order of 100 m or more in only a few months). Constant-thickness models are not capable of describing this movement. Reduced-gravity models are a direct extension of the constant-thickness models that do allow the thickness of the layer to vary. They assume that the ocean consists of a thin surface layer of density \( \rho \) overlying an infinitely deep lower layer of density \( \rho + \Delta \rho \). The depth of the interface between the layers simulates the movement of the ocean pycnocline.
An example of a linear reduced-gravity system is just Equations (1)-(3), except that in (3)

\[ w = -h_t \quad \text{at} \quad z = -H, \quad (6) \]

where \( h_t \) is the instantaneous thickness of the surface layer. A final equation, relating \( h_t \) to pressure, is

\[ h = H + p/g', \quad (7) \]

where \( g' = (\Delta \rho / \rho)g \); this relation follows from the assumption that there are no pressure gradients in the deep ocean. It is only in (7), through its dependence on \( \Delta \rho \), that the deep ocean influences the layer.

Again it is possible to solve for the depth-averaged flow in the layer separately from the shear flow. The depth-independent equations are

\[ u_t - f v + p_x = \tau^x/H + v_h \nabla^2 u, \]
\[ v_t + f u + p_y = \tau^y/H + v_h \nabla^2 v, \]
\[ h_t/H + u_x + v_y = 0, \quad (8) \]

where overbars have been dropped for convenience. The shear-flow equations are Equations (5) with the replacement

\[ u'_x + v'_y + w_z = h_t/H. \quad (9) \]

The shear-flow component of reduced-gravity models is almost always ignored, and Equations (8) by themselves are referred to as the reduced-gravity equations. This neglect amounts to assuming that \( v \) is infinite in the surface layer. [For example, (34) is a solution to (5) for a switched-on wind. Note that \( u' \) and \( v' \) in (34) vanish as \( v \) tends to infinity.]

The modification (9) does not affect the shear flow, \( u' \) and \( v' \), at all; only \( w \) is changed by an additional part that varies linearly from zero at the ocean surface to \(-h_t\) at \( z = -H \). The presence of the term \( h_t/H \) in (8), however, changes the dynamics of the depth-independent component significantly. As a result, Equations (8) are essentially the same as Equations (20) describing one of the baroclinic modes of a continuously stratified ocean, where the separation constant \( c_n \) is given by \((g'H)^{1/2}\).

Several studies include nonlinear terms in reduced-gravity models. Often nonlinear terms involving only depth-independent quantities are added to the depth-independent equations (8). These terms generally do not affect solutions very much, primarily because \( \bar{u} \) and \( \bar{v} \) are typically not sufficiently large. Cane (1979a,b) developed a nonlinear, reduced-gravity system that included all the nonlinear terms. His model is a nonlinear version of the reduced-gravity system described above, except that it is formulated with only two degrees of freedom in the vertical. (The surface layer is divided into
two sublayers. The upper sublayer has a constant thickness and is directly acted upon by the wind.) The nonlinearities affect the solutions significantly in this model, because they involve both the shear and depth-averaged components of the flow field (see the discussion in Section 4.1).

**THERMODYNAMIC, MIXED-LAYER MODELS** Large and persistent SST anomalies often occur in the tropical oceans, a well-known example being the El Niño phenomenon in the Pacific Ocean. The models discussed above are not capable of studying SST anomalies, since the temperature of the layer is fixed. Several reduced-gravity models have recently been developed to overcome this deficiency. They include active thermodynamics, thereby allowing the temperature of the layer to vary. They also allow various processes (the terms, \( w_e \) and \( \bar{w} \), defined below) to change the mass of the layer.

The thermodynamic model of Anderson & McCreary (1984) is a nonlinear extension of Equations (8) that also includes an additional equation for the heat content of the layer. The equations governing the mass and heat content of the layer are

\[
\begin{align*}
h_t + (uh)_x + (vh)_y &= w_e - \bar{w}, \\
(hT)_t + (uhT)_x + (vhT)_y &= \frac{Q_s}{\rho c_p} + w_e T_e - \bar{w} T,
\end{align*}
\]

where \( w_e \) is an entrainment velocity given by

\[
h\Delta T w_e = 2\delta - h \frac{Q_s}{\rho c_p},
\]

\( T \) is the temperature of the layer, \( T_e \) is the constant temperature of the deep ocean, and \( \Delta T = T - T_e \). The system involves three thermodynamic processes: \( Q_s \), \( \delta \), and \( \bar{w} \), where \( Q_s \) is a heat flux into the layer and is given by

\[
Q_s = -\rho c_p \gamma (T - T^*),
\]

\( \delta \) is the rate of potential energy increase due to mechanical stirring by the wind, and \( \bar{w} \) is a slow upwelling velocity in the deep ocean that is presumably driven by the background thermohaline circulation. All three processes are required in order for the layer to maintain itself: \( Q_s \) is necessary to keep the surface layer warmer than the deep ocean, \( \delta \) always acts to increase the thickness of the layer by forcing entrainment of fluid from the deep ocean, and \( \bar{w} \) prevents the layer thickness from increasing indefinitely by continually removing mass from the layer.

Schopf & Cane (1983) developed a thermodynamic, reduced-gravity model involving two active layers, each of which is described by a set of
equations similar to (8) and (10). (The model is actually a bit more complicated than this, because the temperature in the second layer is assumed to vary linearly with depth; this extra degree of freedom requires an additional equation to determine the temperature at the base of the second layer.) Their system allows mass and heat to mix between the layers in a manner similar to the Anderson & McCreary model. The equations for the mass and heat content in the upper layer of their model are

\[ h_t + (uh)_x + (vh)_y = w_e, \]

\[ (hT)_t + (uhT)_x + (vhT)_y = \frac{Q_s}{\rho c_p} + w_e \{ \theta(w_e)T_e + \theta(-w_e)T \}, \]  

where \( w_e \) is determined by the expression

\[ h\Delta T w_e \theta(w_e) = 2\delta - h \frac{Q_s}{\rho c_p} - 2\varepsilon_0 h, \]

where \( \varepsilon_0 \) is a dissipation term for potential energy, \( T_e \) is now the temperature at the top of the second layer and is not constant, and \( \theta(\xi) \) is a step function.

It is instructive to contrast the mixing processes involved in these two models. The way each of them develops and maintains SST anomalies depends crucially on the parameterization of these processes. Some of the similarities and differences are evident from a direct comparison of Equations (10) and (11) with (13) and (14). For example, during entrainment \((w_e > 0)\) the two systems are quite similar, with \( \tilde{w} \) and \( \varepsilon_0 h \) playing analogous roles. During detrainment, however, they differ considerably because of the presence of step functions in (13) and (14). In addition, the two models have quite different states of rest, states in which the right-hand sides of (10) and (13) are set equal to zero.

### 2.2 Linear, Continuously Stratified Models

Layer models are limited because equatorial currents are not entirely confined to the surface mixed layer(s) of the ocean. A logical extension of Equations (1) is to include the effects of continuous stratification and to solve for the flow field throughout the water column. The following set of linear equations is such an extension. Under the conditions specified below, it is possible to find analytic solutions to this set. Primarily for this reason, the solutions of this system provide a foundation for much of our understanding of equatorial dynamics.

In a state of no motion, the model ocean has a stably stratified background density structure \( \rho_b(\xi) \) and an associated Väisälä frequency...
A set of equations linearized about this background state is

\begin{align}
\frac{\partial u}{\partial t} - fu' + p_x &= F - ru' + (v u_z)_z + v_h \nabla^2 u, \\
\frac{\partial v}{\partial t} + fu' + p_y &= G - rv' + (v v_z)_z + v_h \nabla^2 v, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
p_z &= -\rho g, \\
\rho_t - g^{-1} N_b^2 \psi &= (\kappa \rho_z)_z - \varepsilon \rho,
\end{align}

where the terms with the coefficients \( r \) and \( \varepsilon \) are Rayleigh friction and Newtonian cooling, respectively. Surface and bottom boundary conditions are again (2) and (3), with conditions (3) now being applied at the ocean bottom at \( z = -D \). In addition, since the system includes the vertical mixing of heat, two conditions involving \( \rho \) must be applied at the ocean surface and bottom. In many calculations, wind stress is assumed to enter the ocean as a body force, \( F \) and \( G \), with the separable form

\begin{align}
F &= \tau(x, y, t)Z(z), \\
G &= \tau'(x, y, t)Z(z),
\end{align}

where \( \int_{-D}^0 Z(z) \, dz = 1 \). In this event, either there is no vertical mixing of momentum in the system, or a no-stress surface condition is adopted. A comprehensive discussion of similar sets of equations can be found in Veronis (1973) and in Moore & Philander (1978).

**VERTICAL MODES** Without vertical mixing it is always possible to represent solutions to (16) as expansions of the vertical normal modes of the system. With vertical mixing this representation is possible only in special cases. One of these cases is considered by Moore (1980) and is discussed later in this section. Another occurs when \( \nu \) and \( \kappa \) are assumed to be inversely proportional to \( N_b^2 \), the form of mixing of heat is modified to be \( (\kappa \rho_z)_z \), and the boundary conditions on \( \rho \) are that \( \kappa \rho = 0 \) at the ocean surface and bottom (McCreary 1980, 1981). The following discussion assumes that these latter conditions hold.

The vertical modes of the system are the eigenfunctions \( \psi_n(z) \), which satisfy the differential equation

\[ \psi_n'' + \frac{2}{c_n^2} \int_{-D}^z \psi_n \, dz, \]

subject to the boundary condition

\[ \frac{1}{c_n^2} \int_{-D}^0 \psi_n \, dz = 0, \]

and they are usually normalized so that \( \psi_n(0) = 1 \). It is convenient to order them so that the eigenvalues \( c_n \) decrease monotonically. The \( n = 0 \)
The eigenfunction is unique in that \( \eta_0 = 0 \); this eigenfunction is the barotropic mode of the system. (If a free-surface condition is adopted, rather than \( w = 0 \), then \( \eta_0 = \sqrt{gD} \).) The remaining eigenfunctions form an infinite set of barotropic modes.

Solutions to (15) are then given by

\[
\begin{align*}
  u &= \sum_{n=0}^{\infty} u_n \psi_n, \\
v &= \sum_{n=0}^{\infty} v_n \psi_n, \\
p &= \sum_{n=0}^{\infty} p_n \psi_n, \\
w &= \sum_{n=0}^{\infty} w_n \int_{-D}^{D} \psi_n \, dz, \\
\rho &= \sum_{n=0}^{\infty} \rho_n \psi_n,
\end{align*}
\]  

(19)

where the expansion coefficients are functions only of \( x, y, \) and \( t \). The equations governing the expansion coefficients are

\[
\begin{align*}
  u_n - f v_n + p_{nx} &= F_n - (\varepsilon + A/c_n^2) u_n + v_h \nabla^2 u_n, \\
v_n + f u_n + p_{ny} &= G_n - (\varepsilon + A/c_n^2) v_n + v_h \nabla^2 v_n, \\
p_{nt}/c_n^2 + u_{nx} + v_{ny} &= -(\varepsilon + A/c_n^2) p_n/c_n^2, \\
\end{align*}
\]

(20)

and also

\[
\begin{align*}
w_n &= (\partial_t - \varepsilon - A/c_n^2) p_n/c_n^2, \\
\rho_n &= -p_w/g.
\end{align*}
\]

(21)

For convenience, these equations assume unit Prandtl numbers, that is, \( \nu = \kappa = A/N^2 \) and \( r = \varepsilon \). When the wind stress enters the ocean as a body force, the forcing of each mode is

\[
\begin{align*}
  F_n &= \tau^x(x, y, t) Z_n \int_{-D}^{D} \psi_n^2 \, dz, \\
  G_n &= \tau^y(x, y, t) Z_n \int_{-D}^{D} \psi_n^2 \, dz,
\end{align*}
\]

(22)

where \( Z_n = \int_{-D}^{D} Z(z) \psi_n \, dz \). When the wind enters the ocean through the surface stress conditions, the only change is that \( Z_n = 1 \) in (22).

Vertical mixing appears in (20) in the form of a simple drag with a drag coefficient \( A/c_n^2 \), and therefore it affects each mode like the terms proportional to \( \varepsilon \). The difference between the two types of mixing is that \( A/c_n^2 \) increases with mode number (eventually increasing like \( n^2 \)), whereas \( \varepsilon \) does not. Consequently, vertical mixing typically does not influence the low-order modes much, but eventually it dominates the dynamics of the high-order modes (see the discussion of Figures 4a and 4b). Rayleigh friction and Newtonian cooling, on the other hand, affect all modes equally.

Models differ in the type of mixing that is present. Lighthill (1969), who first used and popularized this approach to study equatorial circulation, found solutions to the inviscid form of (15). Many of the models discussed in the next sections are also inviscid. McCreary (1981, 1984) included vertical mixing and in the latter model added horizontal mixing with the simple
form $v_nv_{xxx}$. Models that solve (20) numerically include all the horizontal mixing terms (McCreary et al. 1984, Gent et al. 1983). The Gent et al. (1983) model used Rayleigh friction and Newtonian cooling rather than vertical mixing.

A limitation of this approach is that $\rho$ must be fixed at the ocean surface. As a result, solutions necessarily cannot generate a responsive SST field. To avoid this difficulty, Rothstein (1983) developed a procedure to replace the surface condition on $\rho$ with one on heat flux. First, he modified the surface condition on $\rho$ from $\kappa\rho = 0$ to $\kappa\rho = \kappa\rho_s$, where $\rho_s(x, y, t)$ is an arbitrary density distribution. This change adds the forcing term $-(g\rho_s/c_n^2)\int_0^D \psi_n^2 dz$ to the third equation of (20). Then he chose $\rho_s$ in such a way that the heat-flux condition, $(\kappa\rho_s)_z = 0$, was approximately satisfied. By iterating the procedure, the surface condition of the approximate solution converged to the desired one.

A SURFACE MIXED LAYER  It is possible to incorporate a constant-thickness surface mixed layer in (15) simply by assuming that $N_b(z) = 0$ for $z > -H$ (McCreary 1981). The eigenfunction problem, as defined in (17) and (18), is still well posed, and so the eigenfunctions as well as the solutions (19) remain well behaved. The effect on the eigenfunctions is only that $\psi_n = 0$ in the layer. Therefore, in the mixed layer $u$, $v$, and $\rho$ are depth independent, $w$ varies linearly, and $\rho$ is identically zero.

Moore (1980) considered an interesting special case of Equations (15) that included a mixed layer like that defined in the preceding paragraph in which $v \neq 0$, but assumed that $v = 0$ in the deeper ocean. Boundary conditions at the bottom of the layer are that stress vanishes and $w$ is continuous across $z = -H$. Thus, his model ocean consists of a viscous, constant-density surface layer overlying an inviscid, stratified, deep ocean.

Moore demonstrated that, as in the linear constant-thickness and reduced-gravity models discussed above, the response of this system can be separated into two components. The first is just the surface-layer shear flow described by (5). The second is the flow described by the inviscid form of (15), in which the wind stress enters the ocean as a body force spread uniformly throughout the layer. The important implication of this model, as for the constant-thickness and reduced-gravity models, is that a shear flow can exist in the surface layer that responds independently from the rest of the flow field.

MERIDIONAL MODES  It is also possible to represent solutions to (15) as expansions in meridional, rather than vertical, modes. This approach, however, suffers from several mathematical difficulties. One of them is that the equations of motion must be Fourier transformed in both space and

time, and it is generally not possible to invert the transforms of the solutions with analytic methods. Another is that it is not easy to impose surface and bottom boundary conditions of the form (2) and (3), because these conditions couple together individual meridional modes (McPhaden 1981). No doubt as a result of these difficulties, only three oceanographic studies have used the approach (Wunsch 1977, Philander 1978a, McPhaden 1981), and each assumes a simple forcing of the form $e^{ikx - i\omega t}$ for which it is not necessary to invert any transforms. Because of its limited applications, this approach is not discussed further in this paper.

2.3 Nonlinear, Continuously Stratified Models

There have been a number of studies of fully nonlinear, continuously stratified ocean models. Equations of motion are

\begin{align*}
\frac{\partial u}{\partial t} + uu_x + vu_y + wu_z - fv + p_x &= (\nu u_z)_x + \nu \nabla^2 u, \\
\frac{\partial v}{\partial t} + uv_x + vv_y + wv_z + fu + p_y &= (\nu v_z)_y + \nu \nabla^2 v, \\
\frac{\partial p}{\partial t} + \rho_x + w\rho_z &= 0, \\
\rho_i + \omega \rho_x + \nu \rho_y + \omega \rho_z &= (\kappa \rho_z)_z + \kappa \nabla^2 \rho,
\end{align*}

(23)

and they are solved subject to boundary conditions (2) and (3) with two additional conditions on $p$. The obvious advantage of these equations is that they provide a way of studying the effects of nonlinearities on equatorial circulation. Their disadvantage is that solutions must be found numerically. As a result, it is sometimes difficult to identify the important processes at work in them.

3. UNFORCED SOLUTIONS

Equatorially trapped waves are a complete set of solutions to the unforced version of (20) that have the form $e^{ikx - i\omega t}$. They are an essential part of the response of virtually all stratified models. Their importance is apparent in that solutions to (20) forced by the wind can be represented entirely as packets of waves [as in the solutions (36) and (39)]. In addition, they are visible in the response of nonlinear models as well. This section first defines these waves and then discusses how they are affected by mixing, boundaries, currents, and nonlinearities.

Many of the quantities involved in the following discussion and in the rest of this paper depend in some way on the eigenvalues $e_n$ and should be labeled with the subscript $n$. For notational simplicity this subscript is deleted, unless confusion might result from its absence.
3.1 Equatorially Trapped Waves

A useful approximation, one that is almost universally adopted in equatorial ocean models, is the equatorial $\beta$-plane approximation, which represents the Coriolis parameter by $\beta y$. With this approximation, free solutions to (20) can be conveniently expressed in terms of a discrete set of parabolic cylinder functions $\phi_{\mu_i}(\eta)$. These eigenfunctions satisfy the equation

$$ (\phi_{\mu_i})_{\eta\eta} - \eta^2 \phi_{\mu_i} = -(2\mu_i + 1) \phi_{\mu_i}, \quad (24) $$

where $\eta = (\beta/c)^{1/2} y \equiv \alpha_0 y$. The length scale $\alpha_0^{-1}$ is the equatorial Rossby radius of deformation associated with a particular vertical mode. The eigenvalues are fixed by requiring that each $\phi_{\mu_i}$ vanish at high latitudes. In this case, the eigenvalues are

$$ \mu_i = l, \quad (25) $$

and the parabolic cylinder functions are the Hermite functions $\phi_l(\eta)$. It is useful to normalize them so that $\int_{-\infty}^{\infty} \phi_l^2 d\eta = 1$.

Inviscid equatorially trapped Rossby and gravity waves are

$$ u_{ij} = A_{ij} \frac{c\alpha_0}{i\omega} \Phi_{ij}^- e^{ik_{ij}x - i\omega t}, $$

$$ v_{ij} = A_{ij} \phi_l(\eta) e^{ik_{ij}x - i\omega t}, $$

$$ p_{ij} = A_{ij} \frac{c^2 \alpha_0}{i\omega} \Phi_{ij}^+ e^{ik_{ij}x - i\omega t}, \quad (26) $$

where

$$ \Phi_{ij}^\pm (\eta) = \left( \frac{l+1}{2} \right)^{1/2} \frac{\phi_{l+1}(\eta)}{ck_{ij}/\omega - 1} \pm \left( \frac{l}{2} \right)^{1/2} \frac{\phi_l(\eta)}{ck_{ij}/\omega + 1}, \quad (27) $$

and $\omega = \sigma$. There are two wave numbers designated by the index $j$, associated with each value of $l$. They are

$$ k_{l(j)} = -\frac{\beta}{2c} \left\{ 1 + \left[ 1 - 4 \frac{\omega^2}{\beta^2} \left( \alpha_l^2 - \frac{\omega^2}{c^2} \right) \right]^{1/2} \right\}, \quad (28) $$

where $\alpha_l^2 = \alpha_0^2(2l+1)$. With this choice, $k_{l1}$ ($k_{l2}$) corresponds to a wave with westward (eastward) group velocity. When $\sigma^2/(\beta c)$ is greater than $l + \frac{1}{2} + \sqrt{l(l+1)}$ or less than $l + \frac{1}{2} - \sqrt{l(l+1)}$, $k_{ij}$ is a real number. The higher-frequency waves are gravity waves, and the lower-frequency ones are Rossby waves. Intermediate-frequency waves have complex wave numbers.
and so decay either to the east or the west. Waves for which \( u_{ij} \) and \( p_{ij} \) are symmetric (antisymmetric) about the equator are referred to as symmetric (antisymmetric) waves; with this definition, waves for \( l \) odd (even) are symmetric (antisymmetric).

Two additional free solutions to (20), designated here by the index \( l = -1 \), have a zero meridional velocity field. The equatorially trapped Kelvin wave is

\[
u_{-12} = p_{-12}/c = A_{-12} \phi_0(\eta) e^{ik_{-12}x - i\omega t}, \quad v_{-12} \equiv 0,
\]

and has the dispersion relation

\[
k_{-12} = \omega/c.
\]

The other wave, labeled the anti-Kelvin wave by Cane & Sarachik (1979), has a meridional structure \( e^{i\eta^2} \) and a dispersion relation \( k_{-11} = -\omega/c \). Because this wave grows indefinitely for large \( \eta \), it is not a physically realistic solution in an unbounded ocean.

When \( l = 0 \), the radical in (28) is a perfect square, and the eastward-propagating wave has the dispersion relation

\[
k_{02} = \omega/c - \beta/\omega.
\]

This wave is usually called the Rossby-gravity wave, since it has characteristics associated with both kinds of waves (see Figure 1); it is also

![Figure 1](image)

**Figure 1** Dispersion curves (28) and (30) for inviscid, equatorially trapped waves. The curves are plotted in dimensionless coordinates, with \( \alpha_0 = (\beta/c)^{1/2} \) and \( \sigma_0 = (\beta c)^{1/2} \). The left panel shows the curves for the Kelvin wave \( l = -1 \), for the Rossby-gravity wave \( l = 0 \), and for the \( l = 1, 2, \) and 3 Rossby and gravity waves when the values of \( k_{ij} \) are real. Some of the curves are labeled with their corresponding values of \( l \). The right panel shows the curves for the \( l = 1 \) Rossby and gravity waves, and it also includes the curves in the region where the values of \( k_{ij} \) are complex; the thin curve indicates the imaginary part of \( k_{ij} \). The crosses indicate resonance points for the Rossby and gravity waves.
sometimes referred to as the Yanai wave. The westward-propagating wave
has the dispersion relation \( k_{01} = -\omega/c \). This root is extraneous because it is
not a possible solution to (20). Note that the term involving \( \phi_{l-1} \) in (27)
is not well defined, since both the numerator and denominator vanish for
\( l = 0 \).

Figure 1 plots the dispersion relations (28), (30), and (31). The left panel
shows the curves for the Kelvin and Rossby-gravity waves, and those for
\( l = 1, 2, \) and 3 Rossby and gravity waves. Note that the Rossby-gravity
curve resembles that for a Rossby wave when \( k \ll 0 \) and that for a gravity
wave when \( k \gg 0 \). The right panel shows the dispersion curves for \( l = 1 \)
Rossby and gravity waves, and also for the waves with complex values of
\( k_{11} \) and \( k_{12} \). Similar complex roots also occur for waves associated with
other values of \( l \).

It is possible to define a set of equatorially trapped waves even when the
system includes vertical mixing, Rayleigh drag, and Newtonian cooling.
According to (20) the waves are just those discussed above, except that
\( \omega = \sigma - i A/c^2 - i \epsilon \). The only effect is that the wave numbers (28) and (30)
now always have an imaginary part, and the waves decay in the direction of
their group velocity. The waves still exist when \( \sigma \to 0 \), in which case they
are purely damped waves.

It is not as easy to define a set of waves when there is horizontal mixing.
McCreary (1984) included horizontal mixing of the form \( v_h v_{xxx} \) in (20). In
this case, the waves have the same horizontal structure as in (26) and (29);
however, the dispersion relation for Rossby and gravity waves is quartic,
rather than quadratic, so that there are four roots \( k_{ij} \) for each value of \( l \).
For more general forms of horizontal mixing, it is no longer possible to express
the waves in terms of parabolic-cylinder functions.

### 3.2 The Long-Wavelength Approximation

At low frequencies the two roots in (28) are approximately given by
\( k_{12} = -\beta/\omega \) and

\[ k_{11} = -\frac{\omega}{c} (2l + 1). \tag{32} \]

The Rossby waves with eastward group velocity have short wavelengths (at
the annual frequency \( -2\pi/k_{12} = 55 \text{ km} \)), and so they are not significantly
excited by geophysically realistic (large-scale) winds. They are only
important near western ocean boundaries, where they superpose to create
narrow, intense western boundary currents. The waves with westward
group velocity have long wavelengths [at the annual frequency and with
\( c = 100 \text{ cm s}^{-1} \), \( -2\pi/k_{11} = 10,000/(2l + 1) \text{ km} \)] and are always an impor-
tant part of the response of the tropical ocean. Note also that the long waves
are nondispersive, a mathematically convenient relation.
A number of models take advantage of these properties by making the long-wavelength approximation. This approximation requires the zonal flow field to be in geostrophic balance; that is, it neglects the time-dependent and friction terms in the second of Equations (20). The effect is that the dispersion relations (28) are replaced by the single root (32), and the short waves are filtered completely out of the system.

3.3 Boundary Effects

Northern and Southern Boundaries In the eastern Atlantic the coast of Africa near 5°N is oriented in an east-west direction and thus provides a northern boundary for equatorial flow. How are equatorially trapped waves affected by such a boundary? Hickie (1977) and Philander (1977) discuss waves in an ocean with a northern boundary located at \( y = y_N > 0 \). Cane & Sarachik (1979) consider the effects on the waves when the basin also has a southern boundary at \( y = y_S < 0 \). At these boundaries meridional flow must vanish, and so Equation (24) must be solved subject to the boundary condition \( \phi_{\mu_i} = 0 \) at \( \eta = \alpha_0y_N, \alpha_0y_S \). As a result, the eigenvalues are no longer integers (exceptions are the special cases \( y_N = 0, y_S = -\infty \) and, equivalently, \( y_N = \infty, y_S = 0 \) for which \( u_1 = 2l + 1 \)), and the horizontal structures of the waves change.

Another effect is that the two waves disallowed in the unbounded ocean are now possible (Cane & Sarachik 1979). Since \( \mu_0 \neq 0 \), it follows that the \( l = 0 \) Rossby wave is no longer extraneous. Provided the basin has both a northern and southern boundary, the anti-Kelvin wave is also well defined everywhere in the basin. Neither of the waves, however, is really an equatorially trapped wave. Each owes its existence to the presence of the boundaries and has its largest amplitude there. This property is particularly evident for a low-frequency \( l = 0 \) Rossby wave when the ocean has a single northern (or southern) boundary that is located more than an equatorial Rossby radius (\( \alpha_0^{-1} \)) from the equator. In this case, the \( l = 0 \) Rossby wave has all the properties of a coastally trapped Kelvin wave (Cane & Sarachik 1979).

Eastern and Western Boundaries Equatorially trapped waves reflect from eastern and western boundaries of the ocean, and these reflected waves can significantly affect the tropical ocean. Moore (1968) showed how to find the reflected waves when the boundaries are simple, meridionally oriented barriers. His method chooses the amplitudes of the reflected waves to ensure that \( u = 0 \) everywhere at the boundary. At an eastern boundary a chain of reflected waves is required that leads toward increasing values of \( l \), whereas at a western boundary the chain leads toward decreasing values. Physically, the reason for this difference is that energy can only propagate poleward (equatorward) along an eastern (western) boundary. [See Moore
MCCLARY & Philander (1978) for a recent comprehensive discussion of this approach.

To illustrate Moore’s method, consider the reflection of an equatorial Kelvin wave [Equation (29)] from an eastern boundary at \( x = 0 \). The reflected waves are the Rossby or gravity waves [Equation (26)] with westward group velocity \( (j = 1) \), since reflected energy must be carried away from the boundary. One term of the \( u \)-field of the \( l = 1 \) wave has the meridional structure \( \phi_0 \). By choosing the amplitude of the wave to be 
\[
A_{1,1} = (i\omega/cx_0)\sqrt{2(ck_{11}/\omega + 1)}A_{-1,2},
\]
this part exactly cancels the \( u \)-field of the Kelvin wave at the boundary. The other part of the \( l = 1 \) wave has the structure \( \phi_2 \), and it is now necessary to eliminate this \( u \)-field by using the \( l = 3 \) wave. It follows that an infinite set of waves is required to reflect the Kelvin wave, and according to (27) their amplitudes satisfy the recursion relations
\[
A_{i+2,1} = \left[\left(ck_{i+2,1}/\omega - 1\right)/(ck_{i1}/\omega + 1)\right] \left[(l + 1)^{1/2}/(l + 2)^{1/2}\right] A_{i1}
\]
for \( l = 1, 3, 5, \ldots \). For sufficiently large values of \( l \) the wave numbers \( k_{11} \) of the reflected waves are all complex (as in the right panel of Figure 1), and so this portion of the reflected wave packet remains trapped to the coast. Moore proved that these trapped waves sum to create a \( \beta \)-plane, coastal Kelvin wave that carries energy poleward at sufficiently high latitudes.

Moore’s method can be applied directly to problems that are periodic in time (Cane & Sarachik 1981, McCreary 1981, 1984). It is also possible to use this approach to find the reflection of wave packets that are not periodic in time. The wave packets are first Fourier-transformed in time, then Moore’s method is used to find the reflected waves associated with each Fourier component, and finally the inverse transform of the solution is evaluated. Anderson & Rowlands (1976) used this procedure to find the reflection of a step-function Kelvin wave [with the zonal structure \( \theta(ct - x) \)] from an eastern boundary. The resulting transforms were very complicated and had to be inverted numerically. McCreary (1976, 1980) and Cane & Sarachik (1977) found the reflection of various wave packets from both eastern and western boundaries, but they adopted the long-wavelength approximation. In this case, the inverse transforms can be easily found analytically.

Cane & Sarachik (1977, 1979) pointed out that at sufficiently low frequencies it is not necessary to use Moore’s method to find the reflection of waves from a western boundary. In this limit, the key simplification is that reflected Rossby waves (with the dispersion relation \( k_{12} = -\beta/\omega \)) are approximately nondivergent, and so their contribution to the reflected wave field can be represented conveniently in terms of a stream function \( \psi \), rather than as an expansion in parabolic cylinder functions. One property of this nondivergent contribution is that it generates no net transport across
the boundary \[ \int_{-\infty}^{\infty} \psi_y \, dy = \psi(\infty) - \psi(-\infty) = 0. \] Consequently, any mass flux into the boundary must be balanced by an outgoing flux associated only with the reflected equatorial Kelvin wave. This condition allows the amplitude of the Kelvin wave to be determined independently from the nondivergent contribution.

Several papers consider reflections of waves from sloping boundaries, i.e. boundaries described by the curve \( y = x^{-\alpha} \). Moore et al. (1981) discuss the reflection of \( l = 1 \) Rossby waves from a sloping western boundary. With \( \alpha = 0 \), the only reflected waves are the symmetric \( l = 1, j = 2 \) Rossby wave and an equatorial Kelvin wave. With \( \alpha \neq 0 \), the antisymmetric Rossby-gravity wave is also a part of the reflection, and for some frequencies it is the dominant part. Cane & Gent (1984) also study the reflection of various Rossby waves from a sloping western boundary, but they restrict their analysis to low-frequency waves in order to take advantage of the simplifications discussed in the preceding paragraph. They find that when the boundary slopes, the amplitude of reflected Kelvin waves varies considerably with frequency and vanishes entirely for some frequencies. Clarke (1983) considers the reflection of Kelvin waves from a sloping eastern boundary, but his solutions are valid only off the equator.

ISLANDS Yoon (1981), Rowlands (1982), and Cane & duPenhoat (1982) have studied how islands affect equatorial waves. The islands in these studies have various simple forms. They all are meridionally oriented barriers (with a finite meridional extent \( L \)) that either are infinitesimally thin (a line island) or have a small zonal extent. Provided that an island is centered near the equator, waves are either transmitted or reflected depending largely on the size of \( L \) relative to \( \alpha_0^{-1} \): when \( L < \alpha_0^{-1} \) the waves are not much affected by an island; when \( L > 2\alpha_0^{-1} \) the islands might as well be barriers of infinite extent, and there is very little transmission past the island. Some solutions are for systems of islands representing various low-latitude island chains (like the Gilbert and Galapagos Islands in the Pacific Ocean, and the Maldives in the Indian Ocean). Individual islands in the chains each satisfy the criterion \( L < \alpha_0^{-1} \). As a result, equatorial waves readily propagate through these island chains with little reflection.

3.4 Interactions With Currents

Several papers have considered the interaction of equatorially trapped waves with a background zonal current \( U(y) \). The models are all reduced-gravity models, like (8), except that they also include interaction terms with \( U(y) \). The presence of the current affects the waves in three different ways. First, it can simply act to modify their dispersion relations and to alter their meridional structures. Second, some waves can become unstable and extract energy from the current by barotropic instability. It is possible to
derive criteria that must hold if the currents can be unstable. One of them is that $\text{Re}(\sigma)/k = U(y)$ somewhere in the flow, where $\text{Re}(\sigma)$ denotes the real part of $\sigma$. Finally, waves can also be absorbed near critical layers where $\sigma/k = U$, thereby contributing some of their energy to the current. At the present time there are no studies that explicitly treat the critical-layer absorption of equatorially trapped waves [although Schopf et al. (1981) discuss other effects of critical layers on their solutions], but there are several papers that discuss the other types of interactions.

In complementary studies, McPhaden & Knox (1979) and Philander (1979) considered the modifications of equatorially trapped waves caused by equatorial currents. Both studies avoided the possibility of unstable waves and critical-layer absorption by restricting their discussion to waves for which $\sigma/k > U(y)$ everywhere. McPhaden & Knox discussed effects of both eastward and westward currents on gravity and Kelvin waves, but they did not consider Rossby waves. The Kelvin waves are hardly affected at all by the currents, but gravity waves are visibly distorted. Philander also included Rossby waves in his discussion, but he restricted $U(y)$ to be a narrow eastward current like the Equatorial Undercurrent. The phase speeds of Rossby waves are significantly reduced by the current. Because the current is narrow, however, the decrease is much smaller than that for a current without shear (that is, for a constant $U$); in particular, the phase speeds never become eastward.

Philander (1978b) studied the effect of the surface currents in the tropical Pacific and Atlantic Oceans on equatorially trapped Rossby waves. He chose for $U(y)$ a profile with westward flow from $3^\circ$S to $3^\circ$N and eastward flow from $3^\circ$N to $10^\circ$N in order to simulate the South Equatorial Current and the North Equatorial Countercurrent. Because of the westward equatorial current, critical layers existed for the westward propagating Rossby waves, and some of them were unstable. Philander limited his discussion to a description of the most unstable waves and suggested that this instability may account for the unstable waves reported by Legeckis (1977) and Legeckis et al. (1983) in the Pacific.

Philander (1976) studied a reduced-gravity model with two active surface layers and included different background zonal currents $U_1(y)$ and $U_2(y)$ in the upper and lower layers, respectively. The upper current $U_1(y)$ was similar to that described in the previous paragraph, and $U_2(y)$ resembled the Equatorial Undercurrent. As a result of the presence of westward flow in $U_1(y)$, the system possessed a barotropically unstable wave. In addition, because the background currents involved vertical shear, baroclinic instability was also possible. Philander concluded, however, that baroclinic instability is not effective near the equator, and that the only tropical current that could be baroclinically unstable is the North Equatorial Current north of $10^\circ$N.
3.5 Nonlinear Effects

There are a few studies that consider the effect of nonlinearities on equatorially trapped waves. They all use reduced-gravity models like (8), except that they include some or all of the nonlinear terms. Boyd (1980a) demonstrated that nonlinearities can cause an equatorial Kelvin wave to break, but he concluded that the breaking of Kelvin waves is not likely to be important in realistic ocean basins. Boyd (1980b) derived an expression for the structure of Rossby solitons, and Kindle (1981, 1983) was able to generate them in a numerical model. Finally, in a series of papers, Ripa (1982, 1983a,b) considered resonant interactions of triads of equatorially trapped waves.

4. SOLUTIONS FOR SWITCHED-ON WINDS

This section discusses the response of various equatorial models to a wind field that switches on at some initial time and thereafter remains steady. [See Weisberg & Tang (1983) for a discussion of the response of a reduced-gravity model to growing and propagating wind fields.] For convenience the wind is assumed to have the separable form

\[ \tau^x = \tau^y = \tau_0 X(x)Y(y)\theta(t), \]  

where \( X(x) \) and \( Y(y) \) are arbitrary functions that are usually zonally and meridionally bounded, and \( \theta(t) \) is a step function in time. Both adjustments to equilibrium and the steady solutions themselves are discussed.

4.1 Solutions to Surface-Layer Models

The solution to (5) without horizontal mixing, subject to boundary conditions (2) and (3), is

\[ u' = \frac{2}{H} \sum_{n=1}^{\infty} \left\{ \frac{f\tau^y + vm^2 \tau^x}{v^2m^4 + f^2} + \text{Re} \left[ \frac{\tau^x - it\tau^y}{-vm^2 + if} e^{ist} e^{-vm^2t} \right] \right\} \times \cos mz \theta(t), \]

\[ v' = \frac{2}{H} \sum_{n=1}^{\infty} \left\{ \frac{-f\tau^x + vm^2 \tau^y}{v^2m^4 + f^2} + \text{Re} \left[ \frac{\tau^y + it\tau^x}{-vm^2 + if} e^{ist} e^{-vm^2t} \right] \right\} \times \cos mz \theta(t), \]

\[ w = - \int_{-H}^{z} (u_x + v_y) \, dz, \]

where \( m = n\pi/H \) (D. W. Moore, private communication). The time, speed, and width scales of the solution are apparent in (34). The flow field adjusts to
equilibrium in a time scale $H^2/(\nu \tau^2)$, reaches speeds of the order of $2\tau_0 H/(\nu \tau^2)$, and has a width scale measured by $\nu \tau^2/(\beta H^2)$. For the parameter values of Figure 2, these scales are 6 days, 50 cm s$^{-1}$ and 100 km, respectively. The solution can respond rapidly to the wind because it is not necessary to move isopycnals in the constant-density layer. Another property of (34) is that it is a local balance, that is, it is directly proportional to the strength of the wind $\tau_0 X(x) Y(y)$.

Recall that (34) is not the total response of the layer. For a constant-thickness model there is also a depth-independent contribution to the flow that responds like a barotropic mode. For reduced-gravity and continuously stratified models, this component adjusts much more slowly to the wind because it involves the displacement of isopycnals; this adjustment is discussed later in this section.

Figures 2a and 2b, adapted from Stommel (1960), show the steady response of (34) for both easterly and southerly winds, respectively. Parameter values are $\nu = 20 \text{ cm}^2 \text{ s}^{-1}$, $\beta = 2 \times 10^{-13} \text{ cm}^{-1} \text{ s}^{-1}$, and $H = 100 \text{ m}$, and the amplitude of the wind is 0.5 dyn cm$^{-2}$. The solutions have a number of features that were first discussed heuristically by Cromwell (1953). For easterly winds there is Ekman divergence at the equator forcing equatorial upwelling and a compensating return flow at depth. There is a surface drift at the equator in the direction of the wind. For southerly winds there is a northward surface drift across the equator. This drift causes a divergence and upwelling of fluid somewhat south of the equator, and a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2a}
\caption{Vertical sections of zonal velocity (left) and meridional circulation (right) for a constant-thickness model forced by an easterly wind with an amplitude of 0.5 dyn cm$^{-2}$. The contour interval is 10 cm s$^{-1}$, there is no zero contour line, and the shaded regions indicate westward flow. Calibration arrows are 0.01 cm s$^{-1}$ and 100 cm s$^{-1}$ in the vertical and horizontal directions, respectively. The solution demonstrates that a strong, ageostrophic shear flow can exist in the equatorial mixed layer. After Stommel (1960).}
\end{figure}
convergence and downwelling of fluid somewhat north of the equator. A limitation of these solutions is that all the currents are ageostrophic. Therefore the model does not provide an adequate explanation of the permanent, geostrophically balanced Equatorial Undercurrent. It does, however, provide a picture of the current structures that can rapidly develop in the surface layer in response to fluctuating winds.

Solutions to surface-layer models are sensitive to the choice of bottom boundary conditions. For example, with no-slip bottom conditions linear solutions do not have an undercurrent, but nonlinear solutions do (Charney 1960, Charney & Spiegel 1971). This result suggests, and has been used to argue, that nonlinearities are necessary for the existence of the Equatorial Undercurrent. A better conclusion is that the no-slip conditions are too stringent.

Solutions are also sensitive to the strength of nonlinearities (Charney 1960, Charney & Spiegel 1971). Cane (1979b) contrasts linear and nonlinear solutions in detail. His linear solutions have structures very similar to those in Figure 2 (except that they have only two degrees of freedom in the vertical), and it is useful to refer to this figure when discussing the changes caused by nonlinearities (also see Figure 5). For easterly winds, the nonlinear terms weaken or eliminate the westward surface drift and strengthen the eastward undercurrent. For westerly winds, they strengthen the eastward surface flow and weaken or eliminate the westward undercurrent. For southerly winds, the surface eastward current north of the equator shifts poleward and strengthens, and the subsurface westward flow shifts equatorward until it overlaps the equator.

Two important nonlinear terms are $\nu u_y$ and $\nu u_z$, i.e. the advection of the zonal current by the meridional circulation. If these terms are sufficiently

---

**Figure 2b** As in Figure 2a, except that the wind is southerly. After Stoßmell (1960).
small, they act only to shift the position of the linear zonal currents and to distort their shape. [Robinson (1966) pointed out this property explicitly by treating these two terms, and others, as perturbations to a system of equations like Charney's (1960) system.] The two terms in Cane's solutions appear to act in this way. To illustrate this point, it is again useful to refer to the linear flow fields in Figure 2. For easterly winds, the convergence of fluid at depth advects eastward momentum toward the equator, thereby intensifying the undercurrent. The equatorial upwelling advects the undercurrent upward, thereby weakening the westward surface drift. For westerly winds, the direction of flow of all the currents in Figure 2a is reversed. In this case, surface convergence of eastward momentum intensifies the surface drift, and equatorial downwelling advects eastward flow downward, thereby weakening or eliminating the undercurrent. For southerly winds, the surface eastward flow is advected farther to the north and is concentrated and intensified. The deep westward flow north of the equator is advected farther to the north until it appears on the equator.

4.2 Solutions to Linear, Continuously Stratified Models

ZONALLY INDEPENDENT SOLUTIONS: THE YOSIDDA JET

Moore & Philander (1978) and Cane & Sarachik (1976) discussed inviscid solutions to (20) when the wind is zonally independent [i.e. $X(x) = 1$ in (33)], and McCreary (1980) found a similar solution when the system included vertical mixing. Provided that the zonal flow is assumed to be in geostrophic balance (the long-wavelength approximation), the solution for zonal winds with vertical mixing is

$$u_n = \left( f v_n + F_n \right) \frac{c^2}{A} \left( 1 - e^{-A/c^2} \right) \theta(t),$$

$$v_n = -\frac{1}{\alpha_0 c} \sum_{l=1}^{\infty} \left[ \eta F \right] \frac{1}{2l+1} \phi_l(\eta) \theta(t),$$

$$p_n = c^2 \alpha_0 v_n \frac{c^2}{A} \left( 1 - e^{-A/c^2} \right) \theta(t).$$

(35)

Far from the equator, Equations (35) describe a meridional Ekman drift, $fv_n + F_n = 0$. Near the equator where the Coriolis force is not sufficient to balance the wind, the meridional drift becomes a swift zonal jet. Note that in the absence of vertical mixing (the limit $A \to 0$), the jet continues to accelerate indefinitely. This flow is one of the distinguishing characteristics of equatorial dynamics and is referred to as the Yoshida jet. As we shall see, a similar equatorial jet still exists even when the wind stress is zonally bounded.
If the assumption of zonal geostrophy is not made, Equations (35) also involve oscillations at a discrete set of frequencies \((\beta \alpha)^{1/2}(2l + 1)\) (Moore & Philander 1978). These oscillations are equatorial inertial oscillations, and they are very similar to inertial oscillations at midlatitudes. Their periods, however, are much longer; for example, with \(c = 100 \text{ cm s}^{-1}\) the period is \(2\pi/(\beta \alpha)^{1/2} = 16\) days. They appear in numerical models as well, and to weaken their effect the wind stress is usually turned on gradually, rather than abruptly as in (33).

**ZONALLY BOUNDED SOLUTIONS: ADJUSTMENT TO EQUILIBRIUM**

It is mathematically difficult to find solutions to (20) when the wind stress field is zonally bounded. The solution proceeds by Fourier transforming the equations, finding the solution for each Fourier component, and finally inverting the transforms. As in the Anderson & Rowlands (1976) study, it is necessary to resort to numerical methods to invert the transforms. When the long-wavelength approximation is adopted, however, it is possible to invert the transforms analytically (Cane & Sarachik 1976, 1977, McCreary 1976, 1980). In this case, the inviscid solution in response to a zonal wind of the form (33) is

\[
\begin{align*}
    u_n &= \tau_{0n} \frac{1}{\beta} \left. Y_{YY} \right|_{-\infty}^{\infty} X \, dx' \, \theta(t) + \left\{ \tau_{0n} \frac{1}{2c} \left. Y_0 \phi_0 \right|_{-\infty}^{\infty} X \, dx' \, \theta(t) \right\} \\
    &- \sum_{l=1}^{\infty} A_l \Phi_l(\eta) \left. \frac{c_0}{\beta^2 \alpha_0^2} \right|_{-\infty}^{\infty} X \left( x' + \frac{\beta}{\alpha_0^2} t \right) \, dx' \, \theta(t) \\
    &+ A_{-1} \phi_0 \int_{-\infty}^{\infty} X(x' - ct) \, dx' \, \theta(t),
\end{align*}
\]

\[
\begin{align*}
    v_n &= -\tau_{0n} \frac{1}{\beta} \left. Y_Y X \theta(t) \right|_{-\infty}^{\infty} X \, dx' \, \theta(t) + \sum_{l=1}^{\infty} A_l \phi_l(\eta) X \left( x + \frac{\beta}{\alpha_0^2} t \right) \, \theta(t),
\end{align*}
\]

\[
\begin{align*}
    p_n &= \tau_{0n} (Y - Y_Y) \left. \left( Y \right|_{-\infty}^{\infty} X \, dx' \, \theta(t) \right) + \left\{ \tau_{0n} \frac{1}{2c} \left. Y_0 \phi_0 \right|_{-\infty}^{\infty} X \, dx' \, \theta(t) \right\} \\
    &- \sum_{l=1}^{\infty} A_l \Phi_l^+ (\eta) \left. \frac{c_0^2 \alpha_0}{\beta ^2 \alpha_0^2} \right|_{-\infty}^{\infty} X \left( x' + \frac{\beta}{\alpha_0^2} t \right) \, dx' \, \theta(t) \\
    &+ cA_{-1} \phi_0 \int_{-\infty}^{\infty} X(x' - ct) \, dx' \, \theta(t),
\end{align*}
\]

where \([\eta Y], [Y_Y], Y\), and \(Y\) are the Hermite-expansion coefficients of \(\eta Y\), \(Y_n\), and \(Y\), respectively, \(\tau_{0n} = \tau_0 / \int_0^\infty \psi_n^2 \, dz\), and

\[
A_l = \tau_{0n} \frac{\alpha_0}{\beta} \left( \left[ \frac{\eta Y}{2l + 1} \right] - \left[ Y_n \right] \right), \quad A_{-1} = \frac{\tau_{0n}}{2c} Y_0,
\]
Figure 3  The time development of the surface pressure field (sea level) in response to a patch of easterly wind that is switched on initially. The response is shown at 1 month (upper left), 3 months (lower left), 13 months (upper right), and 5 yr (lower right). The horizontal structure of the wind is indicated in the upper left panel, and the maximum value of the wind stress is 0.5 dyn cm$^{-2}$. The contour interval is 10 cm, there is no zero contour line, and the shaded regions indicate a drop in sea level. The solution illustrates the adjustment of a baroclinic mode to Sverdrup balance. When the wind change is reversed, aspects of the solution compare favorably with sea-level observations during El Niño events. After McCreary & Anderson (1984).

are the amplitudes of packets of equatorially trapped Rossby and Kelvin waves (McCreary 1980). As in (35), there are no inertial oscillations because the long-wavelength approximation is adopted.

It is useful to trace the development of (36) in time. Initially the solutions behave just like the inviscid version of (35). For example, it is possible to show that at small times both $u_n$ and $p_n$ grow linearly in time. At larger times a packet of equatorially trapped waves radiates from the patch. After the passage of these waves, the solution is left in an equilibrium state very different from that in (35).
Except for the terms in braces, the steady part of (36) describes a baroclinic mode in a state of Sverdrup balance, that is, a solution to the steady, inviscid version of Equations (20). (The concept of Sverdrup balance is usually used to describe depth-averaged properties of ocean circulation. The usage is extended in equatorial dynamics to apply also to circulation associated with individual baroclinic modes.) The terms in braces describe a steady, zonally independent equatorial jet, which is the analogue of the growing, inviscid Yoshida jet in (35). The jet does not accelerate indefinitely because the radiation of equatorially trapped waves allows the system to develop a zonal pressure gradient at the equator to balance the wind there. In a bounded basin the equatorial jet is not possible, since there can be no flow through the boundaries, and the steady solution is just Sverdrup flow. Finally, note that the solution is nonlocal in that it extends well beyond the zonal limits of the wind, the equatorial jet being an obvious example.

Figure 3, taken from McCreary & Anderson (1984), illustrates the role of
equatorially trapped waves in the adjustment to Sverdrup balance. The figure shows the response of the surface pressure field when Equations (20) are forced by a patch of zonal wind symmetric about the equator. After one month, equatorial Kelvin waves have propagated into the eastern ocean, but the slower Rossby waves have not yet progressed very far into the western ocean. After three months the equatorial jet is well developed in the central and eastern ocean, as evidenced by the presence of strong meridional pressure gradients there. In the central ocean a zonal pressure gradient develops to balance the easterly wind. In the far eastern ocean the equatorial Kelvin waves have reflected from the eastern boundary as a packet of Rossby waves, and coastal Kelvin waves are also visible, carrying the reflection poleward along the boundary. The reflected Rossby waves begin to eliminate the equatorial jet, as evidenced by the absence of meridional pressure gradients near the eastern boundary. After 13 months the equatorial ocean is essentially in equilibrium with the wind. A measure of the spin-up time of the equatorial ocean, $T$, is the time it takes a Kelvin wave to propagate across the basin and a reflected $l = 1$ Rossby wave to return (Cane & Sarachik 1977). For this solution, $d = 10,000$ km and $c = 125$ cm s$^{-1}$, so that $T = 4(d/c) = 1$ yr. After five years the state of Sverdrup balance is well established in the tropical ocean. Rossby waves, propagating slowly across the ocean, are apparent only well off the equator.

The presence of vertical mixing in the model can change the equilibrium response markedly. One effect is that the propagating radiation field in (36) decays as it propagates with an $e$-folding time scale of $c^2/A$. The steady part of (36) is also changed. When mixing is weak, the steady solution still resembles Sverdrup flow, except that now currents decay away from the region of the wind. For example, to the east of the wind patch the equatorial jet decays with an $e$-folding scale of $c^3/A$. This effect is very important because it means that reflected waves from boundaries can no longer completely cancel the strong equatorial jet; as a result, the addition of mixing actually can strengthen equatorial currents (McCreary 1981). When mixing is strong, steady solutions do not resemble Sverdrup flow at all; instead, they have a steady balance that resembles (35).

**STEADY SOLUTIONS** Figure 4a, taken from McCreary (1981), and Figure 4b show the equatorial circulation produced by (19), (20), and (21) when the ocean is forced by a patch of easterly and southerly winds, respectively, each with an amplitude of 0.5 dyn cm$^{-2}$. The sums in (19) are truncated after $n = 100$, and this number of modes is more than sufficient to ensure that the solution is well converged. The background density field has a strong pycnocline just beneath a surface mixed layer of constant density; the thickness of the layer is 75 m in Figure 4a and 50 m in Figure 4b. There is vertical mixing in the system of the form $v = A/N^2$, and the value of $A$ is set
Figure 4a  Vertical sections of zonal velocity (left) and meridional circulation (right) at the center of the ocean basin for a linear, continuously stratified model forced by an easterly wind. The positions of eastern and western boundaries and the horizontal structure of the wind are similar to those in Figure 3, and the amplitude of the wind stress is 0.5 dyn cm\(^{-2}\). The contour interval is 20 cm s\(^{-1}\), the dashed contours are \(\pm 10\) cm s\(^{-1}\), there is no zero contour line, and the shaded regions indicate westward flow. Calibration arrows are 0.005 cm s\(^{-1}\) and 10 cm s\(^{-1}\) in the vertical and horizontal directions, respectively. There is a geostrophically balanced Equatorial Undercurrent located in the pycnocline. After McCreary (1981).

so that \(v\) has a minimum value of 0.55 cm\(^2\) s\(^{-1}\). As in Figures 2a and 2b, the flow fields have realistic features. For easterly winds there is Ekman divergence, equatorial upwelling, and a compensating return flow at, or slightly above, the level of the core of the undercurrent. The undercurrent is located in the pycnocline and is in geostrophic balance, in good agreement

Figure 4b  As in Figure 4a, except that the ocean is forced by a southerly wind, and the contour interval is 5 cm s\(^{-1}\). Similar to Rothstein (1983).
with the observations. For southerly winds there is northward drift across
the equator with upwelling (downwelling) somewhat south (north) of the
equator. There is a surface eastward current north of the equator, and a
subsurface one to the south.

It is useful to discuss the dynamics of these solutions in terms of the
steady balances of individual modes. Recall that the drag coefficients
associated with vertical mixing, $A/c^2$, increase rapidly with mode number.
For the low-order modes $A/c^2$ is small, and these modes nearly adjust to
Sverdrup balance. They develop pressure gradients that tend to balance the
wind, thereby limiting the strength of the equatorial currents. For the high­
order modes $A/c^2$ is large, and these modes adjust to the steady two­
dimensional balance in (35). They add up to produce the meridional
circulation patterns in Figures 4a and 4b. The equatorial jet is strong only
for intermediate modes (neither boundary waves nor damping significantly
weakens it), and these modes add up to generate the Equatorial
Undercurrent. The change in the character of the steady balances of
individual modes is necessary in order to produce realistic equatorial flow
fields. Solutions in which all the modes are either in Sverdrup balance or in
two-dimensional balance are unrealistic (McCreary 1981).

As noted in Section 2.2, McPhaden (1981) found steady solutions to (15)
by representing them as expansions in meridional modes. He assumed a
steady zonal wind with the horizontal structure $e^{ikx}$ and included in (15)
only mixing of heat and momentum of the form $(vu_z)_z$, $(vv_z)_z$, and $\varepsilon p$. These
restrictions were all necessary in order to be able to represent solutions as
expansions in meridional modes. Solutions developed an undercurrent,
similar to that in Figure 4a.

An advantage of McPhaden’s approach is that it shows clearly how
midlatitude Ekman dynamics change near the equator. There is a vertical­
structure equation associated with each meridional mode that has six
independent wave solutions. For high-order meridional modes (that
contribute to the solution well off the equator), four of the solutions
produce midlatitude Ekman layers at the top and bottom of the ocean, and
the other two produce a geostrophically balanced, midlatitude Rossby
wave. As the meridional mode number decreases, these six solutions
gradually change their character. The four Ekman solutions are increas­
ingly modified by pressure gradients, and the two Rossby waves become
increasingly ageostrophic.

Rothstein (1983) found solutions that were relevant to the circulation in
the eastern Pacific. He drove his model with an idealized representation of
the southeast trade winds; the winds shifted gradually from primarily
easterly in the central and western oceans to southerly near the eastern
boundary. Well away from the eastern boundary, the solution had a strong
Equatorial Undercurrent, much like that in Figure 4a. Near the eastern boundary, the Undercurrent weakened and moved south of the equator. At the eastern boundary, part of the Equatorial Undercurrent joined with a coastal undercurrent south of the equator. In response to equatorial and coastal upwelling, SST was cool along the equator in the central and eastern oceans and along the southern coast.

As mentioned in the Introduction, subsurface eastward flow is not entirely confined to the region of the Equatorial Undercurrent; subsurface countercurrents also exist somewhat off the equator and at greater depths (Tsuchiya 1972, 1975, Cochrane et al. 1979). McPhaden (1984) pointed out that deep eastward currents, similar to these subsurface countercurrents, also exist in several equatorial models (McPhaden 1981, McCreary 1981, Philander & Pacanowski 1980); for example, they are visible on either side of the equator in Figures 4a and 5a. McPhaden found a steady solution and discussed the dynamics of its deep eastward currents in detail.

### 4.3 Solutions to Nonlinear, Continuously Stratified Models

Solutions to nonlinear models like (23) are still sensitive to the strength of vertical and horizontal mixing (Semtner 1981). Semtner & Holland (1980) minimized the effect of mixing in their model by choosing $\nu = \kappa = 1.5 \text{ cm}^2 \text{s}^{-1}$ and by adopting a biharmonic form of horizontal mixing. Their solutions rapidly became unstable, with the Equatorial Undercurrent being barotropically unstable and the westward surface currents somewhat off the equator being baroclinically unstable (Philander 1976). Cox (1980) used somewhat larger values for mixing parameters. His solutions were still unstable, but the eddy activity was considerably less. Philander and coworkers typically use strong mixing in their models; for example, the solutions in Figures 5a and 5b have $\nu = 10 \text{ cm}^2 \text{s}^{-1}, \kappa = 1 \text{ cm}^2 \text{s}^{-1}, v_h = 10^7 \text{ cm}^2 \text{s}^{-1}$, and $\kappa_h = 2 \times 10^7 \text{ cm}^2 \text{s}^{-1}$. With these choices their solutions remain stable, and the effect of the nonlinear terms in their models is primarily to distort the linear solutions in recognizable ways. As noted by Semtner (1981), the observed Equatorial Undercurrent is more stable than the one produced in the Semtner & Holland model, suggesting that the stronger mixing used in other models is more realistic. He concluded that additional studies were needed to explore further the sensitivity of numerical models to the strength and form of mixing. One such study is that of Pacanowski & Philander (1981), which examines the use of vertical-mixing coefficients that depend on Richardson number.

Figure 5a, taken from Philander & Pacanowski (1980), and Figure 5b, taken from Philander & Delecluse (1983), contrast steady linear and
Figure 5a  Vertical sections of zonal velocity at the center of the ocean basin for a nonlinear, continuously stratified model forced by a zonal wind with an amplitude of 0.5 dyn cm$^{-2}$. The response is shown for easterly winds without nonlinearities (left), for easterly winds with nonlinearities (middle), and for westerly winds with nonlinearities (right). The contour interval is 10 cm s$^{-1}$, there is no zero contour, and the shaded regions indicate westward flow. An important nonlinear effect is the advection of the zonal current by the meridional circulation. After Philander & Pacanowski (1980).

Figure 5b  As in Figure 5a, except that the ocean is forced by a southerly wind and the contour interval is 5 cm s$^{-1}$. The response is shown without nonlinearities (left) and with nonlinearities (right). Again, the advection of the zonal currents by the meridional circulation is apparent. After Philander & Delecluse (1983).
nonlinear solutions in response to zonal and meridional winds, respect­
ively, each with an amplitude of 0.5 dyn cm$^{-2}$. The initial density field of the
models has a sharp near-surface pycnocline with a maximum density
gradient at $z = -100$ m, and density is roughly constant for $z > -50$ m.
The flow fields of both linear solutions are similar to those in Figures 4a
and 4b. There is, however, considerable vertical shear in the surface layer
($z > -50$ m), so that these solutions have characteristics of those in both
Figures 2 and 4. Just as for the surface-layer models, it is visibly appar­
tant that important nonlinear effects are the advection of the zonal flow
field by the meridional and vertical currents.

5. SOLUTIONS FOR PERIODIC WINDS

This section discusses the response of various equatorial models to periodic
winds. For most of the solutions, the wind is again assumed to have a simple
separable form

$$\tau^x = \tau^y = \tau_0 X(x) Y(y) e^{-i\sigma t}. \quad (38)$$

The first part of this section describes some important properties of
solutions to linear, continuously stratified models. The concepts of
equatorial resonances, focal points, and beams are discussed. The second
part describes several solutions to periodically forced models of various
types.

5.1 Solutions to Linear, Continuously Stratified Models

The solution to (20) without horizontal mixing and when $X(x)$ is a zonally
bounded function is

$$u_n = \sum_{l=0}^{\infty} \sum_{j=1}^{2} \frac{c\alpha_0}{i\omega} (A_{lj} + R_{lj}) \phi_{lj}^- e^{ik_{lj}x - i\sigma t} + (A_{-12} + R_{-12}) \phi_0 e^{ik_{-12}x - i\sigma t},$$

$$v_n = \sum_{l=0}^{\infty} \sum_{j=1}^{2} (A_{lj} + R_{lj}) \phi_l e^{ik_{lj}x - i\sigma t},$$

$$p_n = \sum_{l=0}^{\infty} \sum_{j=1}^{2} \frac{c^2\alpha_0}{i\omega} (A_{lj} + R_{lj}) \phi_{lj}^+ e^{ik_{lj}x - i\sigma t} + c(A_{-12} + R_{-12}) \phi_0 e^{ik_{-12}x - i\sigma t}$$

(McCreary 1980, 1981). The amplitudes are

$$A_{lj} = \frac{\tau_0 x_0}{i} \frac{(1/c)[\eta Y]_l - (k_{lj}/\omega) [Y_x]_l}{k_{lj} - k_{lj}'} \int_{L_j}^{x} e^{-ik_{lj}x'} dx',$$

$$A_{-12} = \frac{\tau_0 n}{2c} Y_0 \int_{-\infty}^{x} e^{-ik_{lj}x'} dx', \quad (40)$$
where \( j \neq j', L_1 = -L_2 = \infty \), and \( \omega = \sigma - iA/c^2 - i\varepsilon \). The quantities \( R_{ij} \) are the amplitudes of waves that are reflected from eastern and western boundaries. The solution (39) is clearly a complicated superposition of all possible equatorially trapped waves.

**EQUATORIAL RESONANCES** In some circumstances the contribution of a single wave dominates all the others in (39). One such circumstance occurs at equatorial resonances. The amplitude of the Rossby and gravity waves in (40) contains the term \( (k_{ijj} - k_{ij})^{-1} \), and this term is large near resonance points where \( \text{Re}(k_{ijj}) = \text{Re}(k_{ij}) \). Damping in the system prevents the term from ever becoming infinitely large by ensuring that the \( k_{ij} \)'s always have an imaginary part. Several resonance points are indicated by crosses in Figure 1.

There has been only one theoretical study of equatorial resonances. Wunsch & Gill (1976) found a solution somewhat like (39). They pointed out that properties of sea-level observations at various islands near the equator were consistent with the presence of resonant equatorially trapped gravity waves. Luther (1980), in a detailed study of island sea-level data, found more evidence of resonant equatorially trapped gravity waves.

**RAY THEORY** Usually a considerable number of waves contribute to the solution (39), and it is not useful to isolate the contribution of a single wave. It is useful, however, to separate out the contributions of less complex pieces of the solution. The solution, (39) and (19), is a double sum of vertical modes (designated by the index \( n \)) and of the various types of waves associated with each mode (designated by the index \( l \)). Thus, the separation can proceed conveniently in two different ways. One way considers the response of an individual baroclinic mode, i.e. it specifies a value for \( n \) and carries out the summation over all values of \( l \). The other considers only the response of waves of a particular type, i.e. it specifies a value for \( l \) and carries out the summation over \( n \). It is possible to use ray theory to predict where energy associated with pieces defined in these two ways will go.

**Focal points** Several studies have considered the response of a single vertical mode to low-frequency forcing by a zonal wind (Schopf et al. 1981, Cane & Sarachik 1981, Cane & Moore 1981). In each of them, equatorial Kelvin waves generated by the wind field over the interior ocean reflect from the eastern boundary as a packet of Rossby waves [the waves with amplitudes \( R_{ij} \) in (39)]. Energy associated with this packet does not propagate entirely westward, but tends to focus back to the equator. This focusing of energy leaves "shadow zones" in the ocean interior off the equator, i.e. regions where energy associated with reflected Rossby waves does not appear. Figure 6, taken from Schopf et al. (1981), nicely illustrates these features.
Schopf et al. (1981) use ray theory to explain these properties as follows. At any locality, assume that solutions to (20) have the form $A(x, y, t) \exp[i(kx + i\phi(y) - i\omega t)$. Then the dispersion relation for long-wavelength Rossby waves is

$$\sigma = \frac{-\beta k}{\beta^2 y^2/c^2 + l^2},$$

(41)

where $l \equiv \phi_y$ is a local meridional wave number that varies with $y$. (In their discussion, Schopf et al. did not make the long-wavelength approximation.)

The slope of ray paths is the ratio of zonal and meridional group velocities, so that

$$\frac{dy}{dx} = \frac{\sigma_i}{\sigma_k} = 2\frac{\sigma}{\beta} \sqrt{-\frac{\beta k}{\sigma} - \frac{\beta^2 y^2}{c^2}}.$$

(42)

Equation (42) has the general solution

$$y = \left(-\frac{c^2 k}{\sigma \beta}\right)^{1/2} \cos \left[2\frac{\sigma}{c} x + \theta_0\right],$$

(43)

where $\theta_0$ is an integration constant.

Now let the eastern ocean boundary be a meridional barrier located at $x = 0$. Exact and numerical solutions indicate that phase is very nearly constant along the coast for the reflected packet of Rossby waves, so that to a good approximation Equation (43) satisfies the boundary condition

Figure 6  Meridional velocity field produced by equatorially trapped Rossby waves with $\sigma = 2\pi \text{yr}^{-1}$ and $c = 300 \text{cm s}^{-1}$ when phase is fixed to be constant along the eastern boundary [that is, $\theta_0 = 0$ in (43)]. The horizontal structure of the flow, rather than its amplitude, is important here. Energy focuses back onto the equator at a distance given approximately by (44). There are shadow zones where Rossby-wave energy does not appear. After Schopf et al. (1981).
394 McCREARY

\( l \equiv \phi_y = 0 \) at \( x = 0 \). According to (41), when \( l = 0 \) the value of \( y \) is \( (c^2 k / \sigma \beta)^{1/2} \), and so it follows that the integration constant is \( \theta_0 = 0 \). With this choice, it is evident that all ray paths converge to the equator at the positions

\[
x_f_i = -\frac{\pi c}{4 \sigma} (2i + 1),
\]

(44)
i = 0, 1, 2, \ldots. When the exact dispersion relation is used instead of (41), these positions are only approximate focal points. Schopf et al. also found the ray paths when the eastern boundary was sloping. In that case \( \theta_0 \) was a function of \( y \), and the focal points were shifted off the equator.

**Beams** Ray theory can also be used to predict where energy associated with waves of a given type will go. In this case, one looks for approximate WKB solutions of the form \( \exp[i k x + i \phi(z) - i \sigma t] \), and thereby defines a local vertical wave number \( m \equiv \phi_z \). The dispersion relations for Kelvin, Rossby-gravity, and long-wavelength Rossby waves are just Equations (30), (31), and (32), with the replacement \( c_n = N_b / |m| \). As a result, slopes of ray paths are

\[
\frac{dz}{dx} = \frac{\sigma_m}{\sigma_k} \approx (2l + 1) \frac{\sigma}{N_b} \frac{|m|}{m}, \quad -\frac{\sigma}{N_b} \frac{|m|}{m}, \quad -\frac{\sigma}{N_b} \frac{|m|}{m},
\]

(45)

for long-wavelength Rossby waves, Rossby-gravity waves, and Kelvin waves, respectively. Thus, when phase propagates upward \( (m > 0) \), energy propagates downward to the west for long-wavelength Rossby waves and downward to the east for Rossby-gravity and Kelvin waves.

Figure 7a, taken from McCreary (1984), shows the response of the ocean when it is forced by a patch of zonal wind oscillating at the annual period. For convenience, in order that ray paths are straight lines, the background Väisälä frequency has a constant value \( N_b = 0.0045 \, s^{-1} \) beneath a surface mixed layer with a thickness of 75 m. The figure shows three sections of zonal velocity along the equator, taken at a time when the wind is most westerly. The ocean is unbounded, has only an eastern boundary, and has both boundaries in the upper, middle, and lower panels, respectively. All three panels clearly indicate that energy propagates into the deep ocean along ray paths. Without boundaries, energy descends into the deep ocean primarily along two beams that propagate east and west of the wind patch. To the east of the patch, the response is a Kelvin beam with the slope \( -\sigma / N_b \), and to the west it is primarily an \( l = 1 \) Rossby beam with the slope \( 3\sigma / N_b \). When there is an eastern boundary, the Kelvin beam reflects from the eastern boundary as a collection of Rossby beams; the \( l = 1, 3, \) and 5 Rossby beams are clearly visible in the middle panel. With both boundaries,
the response in the deep ocean is complicated because of the multiple reflection of the equatorial beams between the boundaries.

It is interesting that the contributions of individual vertical modes to the bounded basin solutions in Figure 7a all exhibit focal points at the positions predicted by (44), as in Figure 6. One might expect that near these focal points a particular baroclinic mode would stand out above all the others in the complete solutions. Curiously, there is no evidence at all of these focal points in Figure 7a. This result suggests that it is not always a useful exercise to isolate the contributions of individual baroclinic modes from the complete response.

Figure 7b, also taken from McCreary (1984), shows the response of the equatorial ocean when it is forced by a patch of meridional wind oscillating at a period of one month. As in Figure 7a, the background Väisälä frequency is constant beneath a surface mixed layer. The figure shows the meridional velocity field at a time when the winds are most southerly. The propagation of a beam of Rossby-gravity waves is evident in the figure. Even though the basin has both an eastern and western boundary, the response is considerably simpler than that in the lower panel of Figure 7a. The reason is that at a period of one month, no Rossby waves are available for the reflection from the eastern boundary. As a result, the beam reflects entirely poleward via β-plane Kelvin waves (Moore 1968; see the discussion in Section 3.3).

The solutions in Figures 7a and 7b assume that \( N_b \) is constant in the deep ocean. When there is a sharp pycnocline, the WKB approximation is not valid for low-order vertical modes, and some energy can reflect off the pycnocline rather than propagate through it (Philander 1978a). Rothstein et al. (1984) studied in detail the reflection of beams from pycnoclines of various sorts, and they found that there is surprisingly little reflection from the pycnocline. For realistic background density structures, nearly all the energy passed through the pycnocline along ray paths, just as predicted by ray theory.

### 5.2 Other Solutions

Wunsch (1977) studied the response of an inviscid, linear, continuously stratified model like (15), except that forcing in the model was conveniently represented as a surface distribution of vertical velocity (presumably driven by the wind). The forcing had the form of a westward-propagating sinusoidal wave with a period of one year, and the ocean basin was assumed to be infinitely deep and horizontally unbounded (although at the end of his paper, Wunsch reported effects introduced by a western ocean boundary). The solutions, in qualitative agreement with the observations, had a rich vertical structure that was narrowly confined to the equator.
Figure 7b  Similar to Figure 7a, except that a section of meridional velocity is shown when the ocean is forced by a meridional wind oscillating at the monthly frequency. The basin has both an eastern and western boundary. The contour interval is 5 cm s$^{-1}$, there is no zero contour, and the shaded region indicates southward flow. A beam of Rossby-gravity waves propagates into the deep ocean with the slope $-\sigma/N_b$. There are no Rossby waves available to reflect this beam from the eastern boundary, and so it reflects entirely poleward as a beam of coastal Kelvin waves. After McCreary (1984).

In apparent contrast to the solutions in Figure 7a, Wunsch’s solutions produced a great deal of energy in the deep ocean, even though the ocean basin lacked an eastern boundary. In fact, there is no contradiction between the two solutions. They differ because the forcing in Wunsch’s model is not limited zonally. In Wunsch’s solutions, energy still propagates into the deep ocean along ray paths that descend into the deep ocean at shallow angles. Therefore, all the energy present in the deep ocean in his solutions was

Figure 7a  Vertical sections of zonal velocity along the equator when a linear, continuously stratified model is forced by a zonal wind oscillating at the annual frequency. The response is shown for an unbounded ocean (top), for an ocean with an eastern boundary (middle), and for an ocean with both an eastern and western boundary (bottom) at a time when the winds are most westerly. The wind stress is confined to the region $-2500 < x < 2500$ km, is essentially independent of $y$, and reaches a maximum amplitude of 0.125 dyn cm$^{-2}$ in the center of the basin. The contour interval is 20 cm s$^{-1}$, there is no zero contour line, and shaded regions indicate westward flow. Dashed contours are $\pm 10$ cm s$^{-1}$ and are included only if they help to identify prominent features of the flow field. A strong beam of Kelvin waves, propagating into the deep ocean with the slope $-\sigma/N_b$, is evident in all the panels. When the ocean has an eastern boundary, the Kelvin beam reflects as a collection of Rossby beams, propagating into the deep ocean with the slopes $(2l+1)\sigma/N_b$. The flow pattern in the lower panel is complicated because of the multiple reflection of beams between both boundaries. After McCreary (1984).
necessarily generated by a remote part of the forcing. (With a western boundary the source of energy in the west was absent. The deep response was thus due entirely to remote forcing in the east.) In the solutions of Figure 7a, the zonal extent of the basin is effectively made much longer by the reflection of beams from basin boundaries.

Busalacchi & Picaut (1983) forced a reduced-gravity model (a single-baroclinic-mode model) with a realistic representation of the annual winds in the tropical Atlantic. They found that the annual sea-level response along the coast of Africa is most strongly influenced by zonal winds west of 10°W. The alongshore component of the wind also affects coastal sea level, but because that component is weak, it does not affect the coastal ocean nearly as much as the remote winds do. According to (44) and with Busalacchi & Picaut's value of $c$ (125 cm s$^{-1}$), a focal point should (just barely) exist in the Atlantic ($x_{f_0} \approx -5000$ km). No focal point is apparent in their solution, however, most likely because the eastern boundary of their basin is a realistic representation of the actual (sloping) coast of Africa.

In a complementary study to that of Busalacchi & Picaut, McCreary et al. (1984) forced a linear, continuously stratified model with an idealized representation of the annual variation of the equatorial trade winds in the western Atlantic. As in the McCreary (1984) study, waves associated with several different modes superposed to form beams that propagated energy vertically as well as horizontally. Along the equator the response was predominantly a beam of equatorial Kelvin waves and a lowest-order ($l = 1$) Rossby beam. The coast of Africa at 5°N provided a northern boundary to the basin. Along this coast the response was primarily a beam of coastal Kelvin waves ($l = 0$ Rossby waves). The solution compared favorably with several observations from the Gulf of Guinea. In particular, it exhibited vertical propagation of phase along the coast of Africa at 5°N, in good agreement with observations discussed by Picaut (1983).

In the tropical Indian Ocean there is a strong semianual cycle to the zonal wind, with strong westerlies occurring during the transition periods between the Northeast and Southwest Monsoons. To investigate the effect of these winds on the ocean, Gent et al. (1983) forced a linear, continuously stratified model with a realistic representation of the semianual cycle of the winds. Their ocean basin had eastern and western boundaries located at the longitudes 43°E and 97°E, but it lacked northern and southern boundaries. Upward phase propagation existed in their solution and compared favorably with observations from the western Indian Ocean discussed by Luyten & Roemmich (1982). In contrast to the solutions in Figure 7a, equatorial beams were not apparent in their solutions; instead, focal points for several of the modes were visible in the complete solution. The discrepancy between this solution and others that exhibit a beamlike
response is unresolved at the present time. The authors suggested that the difference is due to the complexity of the horizontal structure of the realistic wind field that drove their model, and that beams will only be apparent when $X(x)$ in (38) has a simple form. A more likely possibility is that the significant mixing in their model affected its response considerably.

Philander & Pacanowski (1981) forced a nonlinear, continuously stratified model with idealized zonal wind fields at various frequencies. The nonlinear terms affect the strong surface currents considerably. One obvious effect was that eastward currents are enhanced over westward currents. This strengthening is probably due to the advection of the near-surface zonal currents by the meridional and vertical flow field, in a manner like that discussed in Section 4 for the steady solutions. In contrast, nonlinearities did not affect the weaker deep currents very much. As in several of the linear models, the deep currents appeared to be dominated by a wind-driven Kelvin beam and an $l = 1$ Rossby beam (McCreary 1984).

Cox (1980) forced a nonlinear, continuously stratified model with a realistic representation of the annual cycle of the Pacific trades. At the time of the year when the surface currents were most intense, they became unstable and began to meander with a wavelength of 1000 km and a period of 1.1 months. The region of unstable currents then acted as a forcing region for waves propagating into the deep ocean. Cox noted the presence of Rossby-gravity waves near a depth of 2000 m at a position 20° east of the forcing region, and he used ray theory to explain how this energy propagated into the deep ocean. Thus, the Cox model generated a Rossby-gravity beam similar to the one shown in Figure 7b.

6. MODELS OF EL NIÑO

Early models of El Niño studied the response of the ocean to a relaxation of an idealized representation of the equatorial trade winds (Godfrey 1975, Hurlburt et al. 1976, McCreary 1976, 1977, O'Brien et al. 1981, Philander 1981). The response in Figure 3 is representative of these solutions, particularly of the solution of McCreary (1977). With the direction of the wind and of the response reversed, the figure shows the time development of surface pressure (sea level) in response to a relaxation of the equatorial trade winds confined to the central Pacific. Shortly after the winds relax, sea level rises rapidly in the eastern ocean (equivalent to a deepening of the pycnocline there) and drops in the western ocean, in association with a massive transfer of warm surface water from the western to the eastern ocean. This large-scale response is very similar to changes in sea level that have been observed to take place during El Niño events (Wyrtki 1977, 1979, 1984).
Current research generally falls into three categories. One type of effort finds solutions to ocean models that are forced by realistic, rather than idealized, winds. The advantage of the use of realistic winds is that solutions can be more closely compared with observations. Another type of research investigates the causes of SST anomalies during El Niño events. These studies use ocean models that include thermodynamics as well as dynamics. A third type studies how the ocean and the atmosphere interact during El Niño, by using coupled ocean-atmosphere models. This section summarizes a few papers that illustrate the direction of current research.

6.1 Solutions for Realistic Winds

Busalacchi & O’Brien (1981) forced a linear, reduced-gravity model with shipboard estimates of winds from the tropical Pacific during the period 1961–70. The correlation between time series of model layer thickness in the eastern ocean, $h_e$, and observed sea level from the eastern Pacific was remarkably high, suggesting that the simple model was correctly representing basic equatorial dynamics. The authors demonstrated that the variability in $h_e$ was due to changes in the remote equatorial winds and involved a large-scale response, much like that in Figure 3. In particular, they showed that a westerly wind anomaly in the western equatorial Pacific was responsible for the onset of the 1965 and 1969 El Niño events.

Busalacchi et al. (1983a,b) extended the above calculations to include winds from the period 1971–78. They compared model layer thickness $h$ with observed sea-level variability from the eastern, western, and central Pacific, and the comparisons were good in all three regions. In the eastern ocean, the strong 1972 and 1976 El Niño events were reproduced. As in the 1963 and 1969 events, they were initiated by a collapse of the trade winds in the western Pacific. In the western Pacific, the 1963, 1969, 1972, and 1976 El Niño events were followed by a large decrease in $h$ and also a drop in observed sea level. The decrease in $h$ was the result of the radiation of Rossby waves from a region of weakened trade winds in the central Pacific, again similar to the response in Figure 3. In the central Pacific, $h$ was influenced both by Kelvin waves and by $l = 1$ Rossby waves.

6.2 Thermodynamic Models

Schopf & Harrison (1983) used a slightly modified version of the Schopf & Cane (1983) thermodynamic, mixed-layer model to study the development of SST anomalies during the onset of El Niño. In a series of three numerical experiments, they demonstrated that the development was sensitive to the initial SST field. In the first experiment, the ocean was initially in a state of rest, with uniformly warm SST everywhere in the basin. In the second, the ocean was initially in balance with mean southerly winds and had cool SST
near the eastern boundary south of the equator. In the third, the ocean was in balance with mean easterly winds, with cool SST along the equator in the eastern and central ocean. In all three experiments, westerly winds were switched on initially in the western part of the basin. As in Figure 3, the radiation of Kelvin waves from the western Pacific and their subsequent reflection from the eastern boundary was the basic dynamical process occurring in the model. Warm anomalies occurred only in the latter two cases, which initially had regions of cool SST. The warming was caused predominantly by advective effects associated with the passage of wave fronts, rather than by mixed-layer processes, a result supported by the studies of Philander (1981) and of Gill (1983b).

Schopf (1983) discussed other solutions to the Schopf & Harrison model. In this study, air temperature was assumed to react rapidly to SST, rather than to be a fixed quantity. The effect was to decrease $\gamma$ in (12) by a factor of 10, thereby increasing the e-folding decay time of SST signals from 50 to 500 days. Solutions developed SST anomalies that extended over a much larger area than those in the Schopf & Harrison study, in better agreement with observations.

The 1982 El Niño event differed from the more common events in several ways. One was that in 1982, warm SST anomalies appeared first along the equator in late summer and spread east, whereas during typical events warm SST anomalies appear first at the coast of South America and spread west. Harrison & Schopf (1984) argued that this difference was due only to the fact that the SST field at the time of the onset (summer) was different in 1982. During the spring, SST is cool only near the coast of South America, whereas in the summer a cool tongue exists along the equator as well. They contrasted two solutions to the Schopf & Harrison model that had different initial SST fields resembling the springtime and summertime situations in the Pacific. Model SST anomalies developed differently for the two cases, in a manner consistent with the observed differences.

Zebiak & Cane (1983) used a thermodynamic, mixed-layer model to follow the development of SST anomalies throughout an El Niño event. They first forced their model with a realistic representation of the seasonal cycle of the Pacific trade winds, and they then altered the wind field to include the wind anomalies associated with El Niño discussed by Rasmussen & Carpenter (1982). They considered the relative importance of various thermodynamic processes throughout the event and concluded that no single process (like horizontal advection) dominated at all times.

6.3 Coupled Ocean-Atmosphere Models

McCreary (1983) and McCreary & Anderson (1984) developed simple, coupled ocean-atmosphere models to study El Niño and the Southern
Oscillation. In both studies, the model ocean was a linear, reduced-gravity system like (8). Thermodynamics in the ocean was highly parameterized, with SST being either warm or cool depending on whether the thickness of the layer \( h \) was greater than or less than an externally specified thickness \( h_c \).

In the former study, the model atmosphere consisted of two patches of zonal wind stress \( \tau_w \) and \( \tau_h \), where \( \tau_w \) was a region of strengthened equatorial trade winds in the central ocean, and \( \tau_h \) was a region of strengthened extra-equatorial trade winds in the eastern ocean. The two wind patches interacted with the ocean according to the ideas of Bjerknes (1966, 1969): when the eastern ocean was cool \( (h < h_c) \), \( \tau_w \) switched on simulating an enhanced Walker circulation, and when the eastern ocean was warm \( (h > h_c) \), \( \tau_h \) switched on simulating an enhanced Hadley circulation. In the latter study, the model atmosphere consisted of two wind patches, \( \tau_w \) and \( \tau_s \), where \( \tau_w \) was the same as above and \( \tau_s \) was an idealized version of the annual cycle of the Pacific trade winds.

The interesting property of these two models is that, for reasonable choices of model parameters, they both oscillate at the long time scales associated with the Southern Oscillation. The dynamics of the oscillations is, however, quite different between the two systems. In the former, it is the time it takes Rossby waves generated by \( \tau_h \) to cross the ocean that sets the oscillation period. In the latter, the coupled system has two states of equilibrium: one with \( \tau_w \) switched on, and the other with \( \tau_w \) switched off. The wind field \( \tau_s \) acts as a "trigger" that switches the system from one near-equilibrium state to another.

The above models are limited in that both ocean thermodynamics and the model atmosphere are highly parameterized. Anderson & McCreary (1984) studied a more sophisticated system that included both active ocean thermodynamics and a dynamical, rather than empirical, atmosphere. The model ocean was the nonlinear, reduced-gravity, thermodynamic model described in Section 2 [see the discussion of Equations (10) and (11)]. The equations of motion of the model atmosphere were

\[
\begin{align*}
ru_a - f v_a + p_{ax} &= 0, \\
rv_a + f u_a + p_{ay} &= 0, \\
\varepsilon p_a + c^2 (u_{ax} + v_{ay}) &= Q.
\end{align*}
\]

These equations describe the response of a single, baroclinic mode of the atmosphere to a source of latent heat \( Q \). Despite its simplicity, this model atmosphere has been successfully used in studies of the tropical wind field associated with El Niño (Gill 1980, Zebiak 1982). Interactions between the ocean and the atmosphere were specified as follows. Wind stress driving the
ocean was given by
\[ \tau_x = \rho_d C_d u_x, \quad \tau_y = \rho_d C_d v_y, \] (47)
and the release of latent heat was related to the temperature of the ocean by
\[ Q = -Q_0 \frac{T - T_c}{T(0) - T_c} \theta(T - T_c), \] (48)
where \( \theta(T - T_c) \) is a step function, and \( T(0) \) is roughly the maximum possible value of \( T \). According to (48), there is release of latent heat to the atmosphere only when the ocean becomes warmer than \( T_c \).

Solutions were found numerically for both cyclic and bounded ocean basins. In both cases, instabilities grew to a finite amplitude and began to propagate slowly eastward at speeds of the order of 10 cm s\(^{-1}\), and these disturbances existed for a wide range of parameters. Interestingly, they did not depend on the presence of the advection terms in the ocean; the response of the system was not appreciably changed when these terms were neglected. The bounded-basin solutions compared favorably with observations of El Niño and the Southern Oscillation in several respects; in particular, they oscillated at long time scales. They failed, however, to simulate the rapid onset of El Niño events.

The studies of Lau (1981) and of Philander et al. (1984) provide some insight into the cause of these disturbances. They both considered coupled systems consisting of a linear, reduced-gravity ocean model like (8) and an atmospheric model like (46). (Lau’s model atmosphere also included time-derivative terms.) Both \( \tau_x \) and \( \tau_y \) were specified according to (47), and \( Q \) was taken to be directly proportional to the layer-thickness anomaly \( h - H \). Lau greatly simplified the system by considering only coupling between equatorially trapped Kelvin waves in the ocean and atmosphere, and he found a dispersion relation for the coupled waves. One of the roots always had a phase speed very close to that of an atmospheric Kelvin wave. The other was an oceanic Kelvin wave for weak coupling. As coupling increased, however, the phase speed of the wave slowed to zero and then became positive imaginary, which indicated that it was now a growing instability. Philander et al. solved their system numerically without resorting to Lau’s simplification. They also found a rapidly growing instability.

The instability in these two models develops in the following way. Suppose initially that \( h - H \) is weakly positive in a patch centered on the equator, so that a \( Q \) exists to drive the atmosphere. This \( Q \) drives westerlies to the west of the patch and easterlies to the east of it. Provided the coupling is sufficiently strong, the effect of this convergent wind field on the ocean is
to pile up more mass in the patch, thereby increasing $h - H$ and $Q$ once again. There is no limit on the size of $h - H$ in the model, and so the instability grows indefinitely. The disturbance in the Anderson & McCreary model is caused by a similar mechanism. It does not grow indefinitely because $Q$ is related to $T$, rather than $h - H$, and ocean thermodynamics prevents $T$ from ever increasing much beyond $T(0)$.

7. SUMMARY AND CONCLUSIONS

Equatorial models range in dynamical complexity from linear, constant-thickness, surface-layer models to fully nonlinear, continuously stratified, numerical models. The simpler models have proven to be very useful for identifying the important processes involved in equatorial dynamics. In particular, under certain conditions it is possible to find analytic solutions to the linear, continuously stratified system [(15)], and these solutions provide a basis for understanding much of equatorial dynamics. A thorough understanding of equatorial dynamics is only possible through a careful comparison of solutions to models of all types.

Equatorially trapped waves are a complete set of functions of the unforced version of (20). Without mixing they are the familiar set of Rossby and gravity waves, the Rossby-gravity wave and the Kelvin wave with the dispersion curves shown in Figure 1. With mixing the waves are damped in the direction of their group velocity. The meridional structures of the waves are modified by the presence of northern and southern boundaries, and two extra boundary-trapped waves are possible in this case. Waves reflect from eastern and western boundaries and from islands. Background zonal currents distort the meridional structure of the waves and can cause them to be unstable. There are, as yet, no studies of the critical-layer absorption of equatorially trapped waves.

The response to a switched-on wind differs considerably depending on the type of model and on the zonal structure of the wind. The shear flow in the surface layer of the ocean rapidly adjusts to equilibrium with the wind in a few days. In response to a zonally independent zonal wind, the inviscid solution to (20) is an accelerating equatorial current (the Yoshida jet). If the wind is zonally bounded, the inviscid solution adjusts to a state of Sverdrup balance plus a steady equatorial jet. Waves reflected from ocean boundaries act to eliminate this jet. The addition of mixing can strengthen equatorial currents by preventing reflected waves from completely canceling the jet.

Provided that mixing is sufficiently strong, solutions to continuously stratified models forced by steady winds compare favorably with observations. They produce quite realistic mixed-layer flows, undercurrents, and...
subsurface countercurrents. A comparison of steady, nonlinear and linear solutions suggests that the effect of nonlinearities is primarily to distort the linear solutions in recognizable ways. For example, an important nonlinear effect is the advection of the zonal currents in the direction of the meridional circulation. For smaller values of mixing parameters, nonlinear solutions do not reach a steady state, but rather they are unstable. If mixing is weak, eddy activity in the solutions is unrealistically large.

The response of a linear, continuously stratified ocean to periodic winds is a complex superposition of all possible equatorially trapped waves. It is not usually useful to isolate the contribution of any single wave; the exception is the case of equatorial resonance. Instead, ray theory is a useful technique for determining where energy associated with packets of waves will go. Under suitable conditions \( \theta_0 = 0 \) in (43), Rossby waves associated with a single vertical mode tend to focus their energy back on the equator. In contrast, waves associated with a single type of wave superpose to form beams that carry energy both vertically and horizontally at slopes given by (45). The propagation of beams into the deep ocean is evident in several linear and nonlinear solutions.

Early models of El Niño considered the response of the ocean to an idealized relaxation of the equatorial trade winds. Current research generally falls into three, not entirely exclusive, categories. One type of research forces ocean models with a realistic representation of the Pacific wind field. One model of this type indicated that a weakening of the trade winds in the far western Pacific initiated several El Niño events. Another demonstrated that the drop in sea level at western Pacific islands during El Niño was caused by the radiation of Rossby waves from the central Pacific. A second type of research uses thermodynamic models to study the generation and decay of SST anomalies during El Niño. Several models indicated that the advection of temperature is important during the onset of El Niño, but that other processes are important later on in the event. A third type of research uses coupled ocean-atmosphere models to investigate the ocean’s role in setting the long time scales associated with El Niño and the Southern Oscillation. Several models do oscillate at long time scales, all of them for different dynamical reasons.

There is room for development in all the branches of equatorial modeling discussed in this review. However, significant advances in equatorial modeling have usually followed the discovery of an unexpected and intriguing phenomenon. A good example is the discovery of deep equatorial currents. There was an immediate need to develop models to explain how energy gets into the deep equatorial ocean. A recent example is the 1982/83 El Niño event. This event was unusual in many ways and has stimulated the
development of both atmospheric and ocean models. A safe prediction is that the next few years will see a rapid growth in the field of tropical ocean-atmosphere interaction.

ACKNOWLEDGMENTS

The writing of this paper was sponsored by the National Science Foundation under grant No. OCE 79-21785 through PEQUOD and under grant No. ATM 82-05491. Necessary computations were performed on the CRAY-1 computer at the National Center for Atmospheric Research. NCAR is supported by the National Science Foundation.

Much of the discussion in Section 4.1, concerning the linear solution (34) and its relevance to surface-layer equatorial flows, is an outgrowth of discussions I had several years ago with Dennis Moore and Mike McPhaden. I am indebted to David Anderson, Neill Cooper, Pijush Kundu, Mike McPhaden, and Lew Rothstein, who suggested several improvements of an earlier version of this review. Finally, the efforts of Kevin Kohler and Kathy Maxson are greatly appreciated; without their assistance, the preparation of the manuscript in a timely manner would not have been possible.

Literature Cited

Legeckis, R., Pichel, W., Nesterzuk, G. 1983. Equatorial long waves in geostationary


Lukas, R., Firing, E. 1984. The annual Rossby wave in the central equatorial Pacific Ocean. Submitted for publication


Philander, S. G. H. 1981. The response of the
equatorial ocean to a relaxation of the trade winds. J. Phys. Oceanogr. 11: 176–89