Tools for Computing Tangent Curves for Linearly Varying Vector Fields over Tetrahedral Domains

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Abstract—We present some very efficient and accurate methods for computing tangent curves for three-dimensional flows. Our methods work directly in physical coordinates, eliminating the usual need to switch back and forth with computational coordinates. Unlike conventional methods, such as Runge-Kutta, for computing tangent curves which give only approximations, our methods produce exact values based upon piece-wise linear variation over a tetrahedralization of the domain of interest. We use barycentric coordinates in order to efficiently track cell-to-cell movement of the tangent curves.

Index Terms—Visualization, flow fields, streamlines, tangent curves, vector fields, phase plane, phase volume, critical points, tetrahedral grids.

1 INTRODUCTION

Tangent curves are well accepted as a useful tool for helping to understand a vector field. They are used to construct topological graphs (see [4] and [7]) and they are often used for interactive interrogation of flow fields. In an earlier paper [7], we developed some very efficient methods for computing two-dimensional tangent curves within the context of piecewise linearly varying vector fields. The purpose of this paper is to present some of our results extending this work to three-dimensional vector fields. We assume that we are given the values of the 3D vector field at the vertices of a tetrahedralization of 3D space. The phrase "unstructured grids" is often associated with this type of data. It might be that the tetrahedralization is a result of decomposing the hexahedral cells of a 3D curvilinear grid into five or six tetrahedra. This approach has been discussed by Kenwright and Lane [5], where they employ a fourth order Runge-Kutta method to compute points on a tangent curve. A similar approach, but using Runge-Kutta methods based upon barycentric coordinates, has been discussed by Ueng et al. [10]. All of these previous methods produce approximations to the tangent curves. In this paper, we develop incremental methods that are exact and which run faster than the previous methods which only produce approximations. We give the development in both barycentric and Cartesian coordinates. The advantage to barycentric coordinates is that it makes it very easy to determine when and where a tangent curve leaves a tetrahedral cell. The disadvantage is that conventional computer graphics devices usually expect graphic primitives in Cartesian coordinates. This leaves the final choice of which coordinates to use dependent upon the details of the particular application. In the next section, we give the mathematical background needed for the development of our methods. This mathematical foundation for 2D is well-known, but the analogous mathematical material for 3D is new to the literature. In Section 3 (and Appendices A and B), the main results of this paper are given in the form of incremental formulas for computing points on a linearly varying 3D tangent curve. Some numerical comparisons are given in Section 4.

2 MATHEMATICAL PRELIMINARIES

The emphasis in this paper is on vector fields over three-dimensional domains and, in particular, tetrahedra. Nevertheless, in this preliminary section, we give the mathematical background for the case of a two-dimensional vector field over a triangular domain. The reason for doing this is efficiency in conveying the main ideas and the essence of our results. A quick overview of the mathematics of the 2D situation establishes a beneficial context for explaining the results for a 3D domain.

A linearly varying vector field over a triangular domain has the representation

\[ V(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + b_1 \\ a_{21}x + a_{22}y + b_2 \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + B. \]

A tangent curve

\[ P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \]

over a triangular cell is defined as

\[ P'(t) = V(P(t)) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + B \]

with the initial conditions.

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The 2x2 matrix $A$ and the 2x1 matrix $B$ for a particular triangle are determined from the values of vector field at the vertices of the triangle.

The general solution of (2.2) is of the form

$$P(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = E_1 \Phi_1(t) + E_2 \Phi_2(t) + C,$$  

(2.3)

where the particular functions $\Phi_1$, $\Phi_2$, and the coefficients $E_1$, $E_2$, and $C$ depend on the eigenvalues of $A$. There are five separate cases that $P(t)$ takes depending on these eigenvalues.

**Case 1) $A$ has two real, nonzero eigenvalues.** $(0 \neq r_1 \neq r_2 \neq 0)$

$$P(t) = E_1 e^{r_1 t} + E_2 e^{r_2 t} + P_c,$$

(2.4)

$$E_1 = \frac{A - r_2 I}{r_1 - r_2} (P_0 - P_c),$$

$$E_2 = \frac{A - r_1 I}{r_2 - r_1} (P_0 - P_c),$$

$$AP_c + B = 0.$$

**Case 2) $A$ has one zero and one nonzero eigenvalues.** $(0 = r_1 \neq r_2 \neq 0)$

$$P(t) = E_1 t + E_2 (e^{r_2 t} - 1) + P_c,$$

$$E_1 = \frac{A - A}{r_2} B,$$

$$E_2 = A \left( P_0 + \frac{P_c}{r_2} \right).$$

**Case 3) $A$ has only a zero eigenvalue.** $(r_1 = 0 \neq r_2)$

$$P(t) = e^{t^2} + P_c,$$

$$E_1 = (AP_b + B),$$

$$E_2 = \frac{AB}{2}.$$

**Case 4) $A$ has single real, nonzero eigenvalue.** $(r_1 = r_2 \neq 0)$

$$P(t) = E_1 e^{r_1 t} + E_2 (e^{r_2 t} + P_c),$$

$$E_1 = P_0 - P_c,$$

$$E_2 = (A - r_2 I)(P_0 - P_c),$$

$$AP_c + B = 0.$$

**Case 5) $A$ has complex eigenvalues.** $(\mu + \lambda i, \mu - \lambda i, \lambda \neq 0)$

$$P(t) = E_1 e^{\lambda t} \cos(\lambda t) + E_2 e^{\lambda t} \sin(\lambda t) + P_c,$$

$$E_1 = P_0 - P_c,$$

$$E_2 = \left( \frac{A - \mu I}{\lambda} \right) (P_0 - P_c),$$

$$A P_c + B = 0.$$
curves of each of the images of Fig. 3 are calculated by using initial values at 10 equally spaced points along a particular edge. Blue lines are tangent curves and red lines are vector values at each vertex. Red circles indicate the critical points.

We now present the analogous mathematics for the tetrahedral domain. The linearly varying vector field is defined as

$$V(P) = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}, \quad i = 1, 2, 3, 4.$$  

Similar to 2D cases, the general solution of this initial value problem is of the form

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = E_1 \Phi_1(t) + E_2 \Phi_2(t) + E_3 \Phi_3(t) + C, \quad (2.7)$$

where the particular function \( \Phi_1, \Phi_2, \Phi_3 \) and the coefficients \( E_1, E_2, E_3, \) and \( C \) depend on the eigenvalues of \( \mathbf{A} \). There are nine separate cases that \( P(t) \) takes depending on these eigenvalues.

**Case 1) A has three real, nonzero eigenvalues.**

$$0 \neq r_1 \neq r_2 \neq r_3 \neq 0$$

$$P(t) = E_1 e^{r_1 t} + E_2 e^{r_2 t} + E_3 e^{r_3 t} + P_c$$

$$E_1 = \frac{(A - r_2 I)(r_3 I - A)(P_0 - P_c)}{(r_1 - r_2)(r_3 - r_1)},$$

$$E_2 = \frac{(r_2 I - A)(r_1 I - 2r_2 I + A)(P_0 - P_c)}{(r_2 - r_1)^2},$$

$$E_3 = \frac{E_3}{(r_2 - r_1)} (P_0 - P_c),$$

$$AP_c + B = 0.$$  

**Case 2) A has one real and two equal, nonzero eigenvalues.**

$$0 \neq r_1 \neq r_2 = r_3 \neq 0$$

$$P(t) = E_1 e^{r_1 t} + E_2 e^{r_2 t} + E_3 e^{r_3 t} + P_c$$

$$E_1 = \frac{(A - r_2 I)^2}{(r_2 - r_1)} (P_0 - P_c),$$

$$E_2 = \frac{(r_1 I - 2r_2 I + A)}{(r_2 - r_1)^2}(P_0 - P_c),$$

$$E_3 = \frac{(A - r_1 I)(A - r_2 I)}{(r_2 - r_1)} (P_0 - P_c),$$

$$AP_c + B = 0.$$  

**Case 3) A has single real, nonzero eigenvalue.**

$$r_1 = r_2 = r_3 \neq 0$$

$$P(t) = E_1 e^{r_1 t} + E_2 e^{r_2 t} + E_3 e^{r_3 t} + P_c$$

$$E_1 = P_0 - P_c,$$

$$E_2 = (A - r_1 I)(P_0 - P_c),$$

$$E_3 = \frac{(A - r_1 I)^2}{2}(P_0 - P_c),$$

$$AP_c + B = 0.$$  

**Case 4) A has one real, nonzero and two complex eigenvalues.**

$$r_1 \neq 0, \mu \pm \lambda I, \lambda \neq 0$$
Fig. 3. Examples of tangent curves for all five cases. (a) Case 1 ($\gamma_1 > 0$, $\gamma_2 < 0$). (b) Case 2 ($\gamma_1 = 0$, $\gamma_2 < 0$). (c) Case 1 ($\gamma_1 \neq \gamma_2 < 0$). (d) Case 4 ($\gamma_1 = \gamma_2 < 0$). (e) Case 5 ($\gamma < 0$, $\lambda \neq 0$). (f) Case 6 ($\gamma_1 = \gamma_2 = 0$).
Fig. 4. Notation and conventions for a tetrahedron.

\[
P(t) = E_1e^{\mu t} + E_2e^{\nu t} \cos(\lambda t) + E_3e^{\mu t} \sin(\lambda t) + P_0.
\]

\[
P_1(t) = (A - \mu I)^2 + \lambda^2 \begin{pmatrix} P_0 - P_1 \\ \mu - r_1 \end{pmatrix},
\]

\[
P_2 = \begin{pmatrix} r_1I - A \end{pmatrix} \begin{pmatrix} r_1I - 2\mu I + A \end{pmatrix} \begin{pmatrix} P_0 - P_2 \\ \mu - r_1 \end{pmatrix},
\]

\[
P_3 = \begin{pmatrix} A^2 + \lambda^2 \end{pmatrix} \begin{pmatrix} P_0 - P_3 \\ \mu - r_1 \end{pmatrix}.
\]

\[
AR_1 + B = 0.
\]

Case 5) A has two real, nonzero and one zero eigenvalues.
\[
(r_1 = 0, 0 \neq r_2 \neq r_3 \neq 0)
\]

\[
P_1(t) = E_1(t + E_2(e^{\nu t} - 1) + E_3(e^{\mu t} - 1) + P_0
\]

\[
E_1 = \left( \begin{array}{c}
\frac{r_1I}{r_2 - r_3} \\
\frac{r_2I}{r_2 - r_3} \\
\frac{r_3I}{r_2 - r_3}
\end{array} \right) \begin{pmatrix} P_0 + B \\ P_0 + B \\ P_0 + B \end{pmatrix},
\]

\[
E_2 = \begin{pmatrix} 2 \right) \begin{pmatrix} (I - A) \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix} + A(R_1 + B) \end{pmatrix}.
\]

Case 6) A has one real, nonzero and two zero eigenvalues.
\[
(r_1 = r_2 = 0, r_3 \neq 0)
\]

\[
P_1(t) = E_1(t + E_2(e^{\nu t} - 1) + E_3(e^{\mu t} - 1) + P_0
\]

\[
E_1 = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix},
\]

\[
E_2 = \begin{pmatrix} 2 \right) \begin{pmatrix} (I - A) \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix} + A(R_1 + B) \end{pmatrix}.
\]

Case 7) A has only a zero eigenvalue. \((r_1 = r_2 = r_3 = 0)\)

\[
P_1(t) = E_1(t + E_2(t) + E_3(t) + P_0
\]

\[
E_1 = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix},
\]

\[
E_2 = \begin{pmatrix} 2 \right) \begin{pmatrix} (I - A) \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix} + A(R_1 + B) \end{pmatrix}.
\]

Case 8) A has two real equal, nonzero and one zero eigenvalues. \((r_1 = 0, 0 = r_3 \neq 0)\)

\[
P_1(t) = E_1(t + E_2(e^{\nu t} - 1) + E_3(e^{\nu t} - 1) + P_0
\]

\[
E_1 = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix},
\]

\[
E_2 = \begin{pmatrix} 2 \right) \begin{pmatrix} (I - A) \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix} + A(R_1 + B) \end{pmatrix}.
\]

Case 9) A has two complex and one zero eigenvalues. \((r_1 = 0, \mu \neq \pm \lambda, \lambda \neq 0)\)

\[
P_1(t) = E_1(t + E_2(e^{\nu t} \cos(\lambda t) - 1) + E_3(e^{\nu t} \sin(\lambda t) + P_0
\]

\[
E_1 = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix},
\]

\[
E_2 = \begin{pmatrix} 2 \right) \begin{pmatrix} (I - A) \begin{pmatrix} P_0 + B \\ P_0 + B \end{pmatrix} + A(R_1 + B) \end{pmatrix}.
\]

In Cases 1, 2, 3, and 4, the matrix A of (2.5) is nonsingular and there is a unique critical point satisfying \(AR_1 + B = 0\). Cases 5, 6, 7, 8, and 9 are degenerate cases which have at
least one zero eigenvalue of \( A \). Fig. 5 is intended to be the three-dimensional analog of Fig. 2. The regions of these degenerate five cases are represented as a plane which has at least one zero eigenvalue. This plane is called zero eigenvalue plane. The regions of five degenerate cases on this plane are analogous to the case regions in the 2D situation as shown in Fig. 2. The interior to surface bounded region is Case 4, except for degenerate subcase of Case 9 on the zero eigenvalue plane. The surface boundary is Case 2, except for degenerate subcases of Case 7 and 8 on the zero eigenvalue plane and, for Case 3, indicated by the curve on the two eigenvalue surfaces. The exterior to surface bounded region is Case 1 except for degenerate subcases of Case 5 and 6 on the zero eigenvalue plane. Fig. 6 shows examples for each of the nine cases.

Fig. 6 is the three-dimensional analog of Fig. 3. We include examples of each of the nine cases.

3 Incremental Methods for Computing Tangent Curves

3.1 Conventional Cartesian Coordinates

In order to display a tangent curve, \( P(t) \), a sequence of points on the curve, \( P(t_i) \), \( i = 1, 2, 3, \ldots \) needs to be computed. The explicit solutions for \( P(t) \) given in the previous section, beginning with (2.8), could be used directly for these calculations, but incremental methods are much more efficient for most of the applications we have in mind. In general, incremental methods for the numerical solution of ODEs are of the form

\[
P(t + \Delta t) = N(t, \Delta t)P(t) + I(\Delta t)
\]

which leads to the expansion

\[
P(t + \Delta t) = P(t) + \frac{P'(t)\Delta t}{2!} + \frac{P''(t)\Delta t^2}{3!} + \ldots
\]

Both of the series for \( P'(\Delta t) \) and \( P''(\Delta t) \) have limits that can be approximately calculated in a variety ways. The simplest one is to truncate the series. If we truncate the series, then we have the same formulas as would be obtained with Euler or Runge-Kutta methods applied to the special case of linearly varying vector fields; namely:

**Euler's:**

\[
P(t + \Delta t) \approx (I + \Delta t \cdot A)P(t) + \Delta t \cdot B
\]

**Runge-Kutta Second Order:**

\[
P(t + \Delta t) \approx \left(I + \Delta t \cdot A + \frac{\Delta t^2}{2} \cdot A^2\right)P(t) + \Delta t \left(I + \frac{\Delta t}{2} \cdot A\right)B
\]
Fig. 6. Examples of tangent curves for all nine cases. (a) Case 1 ($r_1 < 0, r_2 \neq r_3 > 0$). (b) Case 2 ($r_1 < 0, r_2 = r_3 > 0$). (c) Case 3 ($r_1 = r_2 = r_3 > 0$). (d) Case 4 ($r_1 > 0, \mu < 0$). (e) Case 5 ($r_1 = 0, r_2 < 0, r_3 > 0$). (f) Case 5 ($r_1 = 0, r_2 \neq r_3 > 0, \mu = 0$).

Runge-Kutta Fourth Order:

\[ P(t + \Delta t) \approx \left( \sum_{i=0}^{4} \frac{(\Delta t \cdot A)^i}{i!} \right) P(t) + \Delta t \left( \sum_{i=0}^{3} \frac{(\Delta t \cdot A)^i}{(i+1)!} \right) B \]

Adaptive Runge-Kutta-Fehlberg:

\[ P(t + \Delta t) \approx \left( \sum_{i=0}^{4} \frac{(\Delta t \cdot A)^i}{i!} \right) P(t) + \Delta t \left( \sum_{i=0}^{3} \frac{(\Delta t \cdot A)^i}{(i+1)!} + \frac{(\Delta t \cdot A)^5}{2080} \right) B \]
\begin{align*}
\text{Error Estimate } & \approx (\Delta t \cdot A)^2 \left( \frac{\Delta t \cdot A}{2080} - \frac{1}{780} \right),
\end{align*}

The above formulas are only approximate solutions for linearly varying vector fields. Based upon the explicit solutions given in the previous section, it is possible to obtain incremental methods which are exact. These methods are of the form

\[ P(t + \Delta t) = G(\Delta t)P(t) + H(\Delta t). \]

The particular representation of \( G(\Delta t) \) and \( H(\Delta t) \) depends upon the properties of the parameters of the
defining differential equation, namely $A$ and $B$ of (2.5). For example, if $A$ has three real, nonzero eigenvalues, $r_1$, $r_2$, and $r_3$, then

$$P(t + \Delta t) = E \left( \begin{array}{ccc} e^{\Delta r_1} & 0 & 0 \\ 0 & e^{\Delta r_2} & 0 \\ 0 & 0 & e^{\Delta r_3} \end{array} \right) E^{-1}(P(t) - P_i) + P_i,$$

(3.1)

where the matrix $E$ is $(E_1 E_2 E_3)$ and $E_i$, $i = 1, 2, 3$ are defined in (2.8).

Equation (3.1) allows for a very efficient computation of the points on the tangent curve. We should also point out that these formulas give exact values and are not approximations as provided by Euler’s method or Runge-Kutta methods. The formula of (3.1) is only one of nine possible cases. The details for all nine cases can be found in Appendix A.

### 3.1.1 Incremental Methods Using Barycentric Coordinates

Within the context of tetrahedral domains, there are some advantages to using barycentric coordinates. One of the main advantages is that it is relatively easy to determine when the curve leaves the tetrahedral domain and, in particular, on which triangular face it actually leaves. This can be coupled with tetrahedral grid information to give the data for the next tetrahedron needed for stepping through a tetrahedralized domain. We include some background material on barycentric coordinates.

Given a point

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the barycentric coordinates, $b_i$, $b_j$, $b_k$, and $b_l$ of this point relative to the tetrahedron $T_{ijkl}$ with vertices $P_i$, $P_j$, $P_k$, and $P_l$ are defined by the relationships

$$P = b_i P_i + b_j P_j + b_k P_k + b_l P_l, \quad b_i + b_j + b_k + b_l = 1.$$ 

For more details on the efficient computation of barycentric coordinates see Nielson [6].

The barycentric coordinates of $(x, y, z)$, $b_i$, $b_j$, $b_k$, and $b_l$ are linear functions of $x$, $y$, and $z$. It is useful to explicitly show this dependency at times and, so, we let the column vector consisting of the four barycentric coordinates be denoted as

$$b(x, y, z) = \begin{pmatrix} b_i(x, y, z) \\ b_j(x, y, z) \\ b_k(x, y, z) \\ b_l(x, y, z) \end{pmatrix}.$$ 

The $4 \times 4$ matrix $V$, whose columns consist of the barycentric coordinates of the four flow vectors at each of vertices, will be denoted by

$$V = \begin{bmatrix} b(V_i + P_i) - b(P_i) b(V_j + P_j) - b(P_j) b(V_k + P_k) - b(P_k) b(V_l + P_l) \end{bmatrix}.$$ 

(3.2)

Linear variation over a tetrahedron can be expressed in term of barycentric coordinates as

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + B = (P_i P_j P_k P_l) V \delta(x, y, z).$$

The matrix $V$ has the same eigenvalues as $A$ with the additional eigenvalue of zero.

The tangent curve $P(t)$ will be represented in terms of barycentric coordinates as

$$p(t) = b(P(t)) = \begin{pmatrix} b_i(t) \\ b_j(t) \\ b_k(t) \\ b_l(t) \end{pmatrix},$$

and the basic differential equation, which characterizes $P(t)$ becomes

$$p'(t) = Vp(t).$$

(3.3)

with initial condition $p(0) = b(P_0) = p_0$.

Similar to Cartesian coordinates, the general solution of (3.3) in terms of barycentric coordinates is of the form

$$p(t) = c_1 \phi_1(t) + c_2 \phi_2(t) + c_3 \phi_3(t) + c_4,$$

where the particular functions $\phi_1, \phi_2, \phi_3$ and the coefficients $c_1, c_2, c_3,$ and $c_4$ are determined by the eigenvalues of $A$.

Incremental methods for barycentric coordinates are based upon formulas which are only slightly different from those for Cartesian coordinates, namely

$$p(t + \Delta t) = g(\Delta t)p(t).$$

(3.4)

We can use repeated application of the basic differential equation in barycentric coordinates to yield

$$p(t + \Delta t) = \left( I + \Delta t V + \frac{(\Delta t V)^2}{2!} + \frac{(\Delta t V)^3}{3!} + \cdots \right)p(t).$$

(3.5)

As before, a truncation of the series of (3.5) leads to approximations equivalent to those obtained by applying Euler or Runge-Kutta methods to linearly varying vector fields. This approach has been documented by Ueng et al. [10] and Kenwright and Lane [5]. Using the explicit solutions of Section 2, we are able to compute explicit representations of the limit of (3.5). As with the case with Cartesian coordinates, there are nine (9) separate cases depending upon the eigenvalues of $A$. For example, if $A$ has three, nonzero eigenvalues, then

$$p(t + \Delta t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{V}{r_1} + \frac{V}{r_2} + \frac{V}{r_3} \left( \begin{array}{c} r_1 I - V \\ r_2 I - V \\ r_3 I - V \end{array} \right) \left( \begin{array}{c} e^{\Delta r_1 t} - 1 \\ e^{\Delta r_2 t} - 1 \\ e^{\Delta r_3 t} - 1 \end{array} \right) + O(\Delta t).$$

Our algorithms use the following general strategy: Once a tangent curve enters a tetrahedron, the matrix coefficients of the scalar values involving $\Delta t$ are computed and stored. This allows for the efficient computation of $g(\Delta t)$ of (3.4) in...
the presence of possible changes to $\Delta t$. The matrix $g(\Delta t)$ is then used to step the tangent curve through the tetrahedron. We continuously monitor each of the four components of $p(t)$. If any one of them becomes negative, then we know that the curve is leaving this tetrahedron. For example, if $h_3$ becomes negative, then the curve is leaving through the triangle face with vertices $V_0$, $V_3$, and $V_4$. In many of our applications, when neighborhood information is available, we can get the data for the next tetrahedron to process in order to continue along the path of the tangent curve.

### 4 Examples

In order to compare our new method with the conventional methods, we include some numerical results. If

$$ P'(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, $$

then the exact solution is a helix circling around the z-axis. As a measure of the error, we take the distance from $P(t)$ to the z-axis. This should be 1.0. In Table 1, we show this error.
for Euler’s Method, the Fourth order Runge-Kutta method, and our new exact method.

5 REMARKS

We did not discuss topological methods here. One of the first steps towards an algorithm for computing the topological graph of a flow field is a means of characterizing the critical points (points where the flow is zero). In the 2D case, the mathematics required to make this analysis is provided by the explicit solutions given in the five cases starting with (2.4). The resulting characterization is often called “phase plane” analysis and, within this context, the explicit solutions for 2D that we have given here are not new, except possibly for the particular form used. But, to our knowledge, the explicit solutions in 3D (for all cases), starting with (2.4), are new to the literature. Here, we have used these representations to obtain efficient incremental methods for computing 3D tangent curves. The next step is to use these explicit solutions to do thorough and complete 3D critical value characterization, including the characterization of points of attachment and detachment on the inner boundary surface (see [7] for some discussion of inner boundary critical points in the 2D situation). We have begun this work and will report upon our findings in a forthcoming report. We should also mention that some work in this direction has been reported by Globus et al. [2] which they attribute to Abraham and Shaw [1].

APPENDIX A

INCREMENTAL EQUATIONS FOR CONVENTIONAL CARTESIAN COORDINATES

The formulas contained in this section were obtained by using the Cayley-Hamilton Theorem (a matrix satisfies its own characteristic equation) and the explicit equations found in Section 2 starting at (2.8). The matrix $[E_1\ E_2\ E_3]$ is denoted by $E$ and the tangent curve $P(t)$ is defined by

$$P(t) = AP(t) + B, \quad P(0) = P_0.$$  

Case 1) $A$ has three real, nonzero eigenvalues. 

If $|E| \neq 0$, then

$$P(t) = \left[ e^{A_1t} \quad 0 \quad 0 \
0 \quad e^{A_2t} \quad 0 \
0 \quad 0 \quad e^{A_3t} \right] E^{-1}(P(t) - P_c).$$

If $|E| = 0$, then the tangent curve will be straight line and, so,

$$P(t) = P(t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 2) $A$ has one real and two equal, nonzero eigenvalues. 

If $|E| \neq 0$, then

$$P(t) = \left[ e^{A_1t} \quad 0 \quad 0 \
0 \quad e^{A_2t} \quad 0 \
0 \quad 0 \quad e^{A_3t} \right] E^{-1}(P(t) - P_c).$$

If $|E| = 0$, then

$$P(t) = P(t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 3) $A$ has single real, nonzero eigenvalue. 

If $|E| = 0$, then

$$P(t + \Delta t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 4) $A$ has one real, nonzero and two complex eigenvalues. 

If $|E| = 0$, then

$$P(t + \Delta t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 5) $A$ has two real, nonzero and one zero eigenvalues. 

If $|E| = 0$, then

$$P(t + \Delta t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 6) $A$ has one real, nonzero and two zero eigenvalues. 

If $|E| = 0$, then

$$P(t + \Delta t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 7) $A$ has only a zero eigenvalue. 

If $|E| = 0$, then

$$P(t + \Delta t) = P(t) \pm \Delta t(P_c - P_0).$$

Case 8) $A$ has two real equal, nonzero and one zero eigenvalue.

If $|E| = 0$, then

$$P(t + \Delta t) = P(t) \pm \Delta t(P_c - P_0).$$
Appendix B: Incremental Equations for Barycentric Coordinates

The equations given here are completely analogous to those of Appendix A, except barycentric coordinates are used in place of Cartesian coordinates. The matrix \( V \) is defined in (3.2).

Case 1) \( A \) has three real, nonzero eigenvalues. (\( \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0 \))

\[
P(t + \Delta t) = \left( I + \frac{V}{r_1} \left( \frac{V - r_2 I}{r_2} \right) \frac{r_3 I - V}{r_3 - r_1} \right) e^{\lambda_1 \Delta t} - 1
\]

Case 2) \( A \) has one real and two equal, nonzero eigenvalues. (\( \lambda_1 \neq r_2 \neq r_3 \neq 0 \))

\[
P(t + \Delta t) = \left( I + \frac{V}{r_1} \left( \frac{V - r_2 I}{r_2} \right) \frac{r_3 I - V}{r_3 - r_1} \right) e^{\lambda_1 \Delta t} - 1
\]

Case 3) \( A \) has single real, nonzero eigenvalue. (\( \lambda_1 = r_2 = r_3 \neq 0 \))

\[
P(t + \Delta t) = \left( I + \frac{V}{r_1} \left( \frac{V - r_2 I}{r_2} \right) \frac{r_3 I - V}{r_3 - r_1} \right) e^{\lambda_1 \Delta t} - 1
\]
Case 9) A has two complex and one zero eigenvalues. 
\( \gamma_1 = 0, \mu \pm \lambda \), \( \lambda \neq 0 \)

\[
p(t + \Delta t) = \left( I + V \Delta t + V^2 \frac{2 \mu V + (3 \mu^2 - 2 \lambda^2)I}{(\mu^2 + \lambda^2)^2} \right) \\
\left( e^{\alpha \Delta t} \cos(\lambda \Delta t) - 1 \right) \\
+ V^2 \frac{2 \mu^2 - \lambda^2 V - \mu(\mu^2 - 3 \lambda^2)I}{\lambda(\mu^2 + \lambda^2)^2} \\
\left( e^{\alpha \Delta t} \sin(\lambda \Delta t) - \lambda \Delta t \right) p(t).
\]

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