A theoretical study of topographic effects on coastal upwelling and cross-shore exchange

Y. Tony Song *, Yi Chao

Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109, USA

Abstract

The effects of topographic variations on coastal upwelling and cross-shore exchange are examined with a theoretical, continuously stratified, three-dimensional coastal ocean model. The model takes into account topographic variations in both alongshore and cross-shore directions and allows analytical solutions with an Ekman surface layer that faithfully represents the physical nature of the coastal upwelling system. Theoretical solutions with any analytical form of alongshore-varying topography can be solved based on the perturbation method of Killworth [J. Phys. Oceanogr. 8 (1978) 188]. Analyses of the model solutions lead to the following conclusions:

(1) The variation of upwelling fronts and currents is shown to be caused by the combined effect of topography and stratification. Topographic variation causes uneven upwelling distribution and leads to density variation, which results in a varying horizontal pressure gradient field that causes the meandering currents. The variation index is dependent upon a bilinear function of their physical parameters—the ratio of the topographic variation depth to the total depth and Burger's number of stratification.

(2) Cross-shore slope is found to play a role in maintaining the meandering structure of the alongshore currents. The anticyclonic circulations can further induce downwelling on the offshore side of the current, while the cyclonic circulations enhance upwelling and form upwelling centers on the inshore side of the current.

(3) Alongshore topography does not change the total upwelled water, i.e., the total Ekman pumping is conserved. However, it increases cross-exchange of water masses by transporting inshore (offshore) water near topographic features far offshore (inshore) from the mean position of the front.

The applicability and limitations of the theory are also discussed.

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*Corresponding author.
E-mail address: song@pacific.jpl.nasa.gov (Y.T. Song).
Wind-driven circulation and Ekman transport are fundamental theories for understanding coastal ocean dynamics (Allen, 1980). In the coastal ocean, topographic effect becomes an important factor because of the shallow water depth, which requires that momentum, energy, and heat be absorbed by a relatively small volume of water. It is necessary to consider the combined effects of wind forcing and topographic variations on coastal oceans for studying cross-shore exchange of water masses, which affect the availability of nutrients, the distribution of sediments, and the transport of natural and anthropogenic products to the deep ocean.

Satellite observations often show multiple upwelling centers distributed along the coast (see Fig. 1), each with a tongue of cold water projecting in the offshore direction and interleaving with a tongue of warmer water towards the coast (Strub et al., 1991). Associated with these upwelling centers are cold filaments in the coastal regions (e.g., off the West Coast of the United States), which transport upwelled, nutrient-rich, water far offshore from the mean position of the fronts. These multiple upwelling centers and cold filaments seem to occur at specific locations every year (e.g., Magnell et al., 1990; Glenn et al., 1996). By examining the sea surface temperature of Fig. 1 and the underlying topography of Fig. 2, we found that the upwelling centers and cold filaments along the central California coast coincide with topographic features, e.g., Pt. Arena, Pt. Reyes, Pt. Ano Nuevo, and Pt. Sur. Baroclinic instability has been the traditional explanation for causing the meandering upwelling fronts to break and form cold filaments (e.g., Ikeda and Emery, 1984; Ikeda et al., 1985).

Fig. 1. Multiple upwelling centers off central California on 12 June 1993 (Courtesy E.M. Armstrong) locate at Pt. Arena, Pt. Reyes, Pt. Ano Nuevo, and Pt. Sur. Each center grows a cold water tongue that projects in the offshore direction and interleaves with a tongue of warmer water moving towards the coast.
Barth, 1989). What is not clear is how the baroclinic instability breaks the upwelling fronts at approximately the same locations. Could the local topography play a role in the formation of the multiple upwelling centers and cold filaments?

The classical theory of shelf circulation is based on the two-dimensional coastal upwelling model (Pedlosky, 1978): upwelling favorable winds (northerly in the northern hemisphere and southerly in the southern hemisphere) drive offshore currents at the surface, and the surface layers are replenished by cold, nutrient-rich water from depth. Equatorward transport is geostrophically balanced by the cross-shore pressure gradient. This model was used to understand the wind-driven coastal upwelling circulation. However, this model does not consider the alongshore-varying features of the upwelling system. Observations show that upwelling fronts can develop strong variations and display huge meanders along the coast, a phenomenon that cannot be explained by

Fig. 2. Coastal bathymetry off central California shows multiple ridges located at Pt. Arena, Pt. Reyes, Pt. Ano Nuevo, and Pt. Sur and interspersed with Bodega Bay, San Francisco Bay, and Monterey Bay. Bathymetry contours above 200 m are shown with 20 m interval.
the two-dimensional cross-shore-section model. Killworth (1978) took the alongshore-varying upwelling fronts into consideration with a theoretical, uniformly stratified, linear model. He examined the effects of small, slow, alongshore-varying topography on upwelling and internal Kelvin wave propagation. His results revealed a fully three-dimensional solution with enhanced upwelling (downwelling) on the south side of a ridge (canyon). Wind stress is represented by a delta function sink of fluid situated at the intersection of the sea surface and the coast, similar to that used by Pedlosky (1978). This assumption greatly simplifies the near-surface physics (e.g., the offshore transport of surface water in the Ekman layer) that is important for studying cross-shore exchange.

A great deal of studies of wind-driven current variability, especially in terms of coastal-trapped-wave theory, have been carried out (e.g., Allen, 1980; Brink, 1986; Chapman, 1987). Two comprehensive reviews on this subject are given by Allen (1980) and by Brink (1991). Laboratory experiments (Narimousa and Maxworthy, 1986) and numerical models (Peffley and O’Brien, 1976; Haidvogel et al., 1991; Song and Haidvogel, 1993) are also used to study the role of bottom topography and coastline curvature in enhancing upwelling. Some studies (Klinck, 1989; Allen, 1996) focus on the topographic effect of narrow submarine canyons. However, there is no single theory that considers both topography and realistic surface Ekman dynamics, primarily because of the mathematical difficulty involved in obtaining analytical solutions.

Recently, Song et al. (2001) included an Ekman transport layer in the model of Killworth (1978) and applied the model to the observed upwelling centers off the New Jersey coast. The Ekman transport layer is parameterized by a vertical shear friction profile and allows model solutions to have unevenly upwelling fronts, similar to the features observed by satellite imageries (Glenn et al., 1996). However, the combined topographic variations in both alongshore and cross-shore directions are neglected in their model, which makes the results less applicable to most coastal oceans. Topographic variation in both directions can be important as shown by Brink (1986) in the case of scattering coastal-trapped waves. The main objective of this study, therefore, is to extend the theoretical models of Killworth (1978) and Song et al. (2001) by allowing topography to vary in both directions and by including the surface Ekman dynamics. The extended model overcomes the mathematical difficulty and allows us to derive new conservation properties of total upwelled water and to have theoretical solutions with topography in both alongshore and cross-shore directions. Such an improved model is necessary to further extend our theoretical understanding of the three-dimensional structure of the upwelling system. One specific focus is on examining the topographic effects on the upwelling fronts and cross-shore exchange of water masses in the coastal regions off central California.

2. The wind-driven, continuously stratified model

2.1. Model geometry and equations

We consider a simple rectangular coastal region (Fig. 3) with one side representing the coastal wall and the opposite side representing the offshore boundary. The other two sides of the region represent the cross-shore open boundaries. The model topography is a simplification of the ridge features as shown in Fig. 2, representing a narrow strip along the outer-shelf off central California.
The topography is allowed to vary in both alongshore and cross-shore directions, an extension of Killworth (1978) and Song et al. (2001),

$$h(x, y) = 1 - B(x, y),$$

where $B(x, y)$ is the topographic variation relative to total depth, 1 (nondimensional units). Using $x, y, z$ as Cartesian coordinates with the $z$-axis pointing vertically and the $x$ and $y$ axes pointing horizontally, the nondimensional primitive equations can be written:

$$\sqrt{\gamma} u_t + \varepsilon \mathbf{v} \cdot \nabla u - v = -P_x + \tau^y F^u(z),$$
$$\sqrt{\gamma} v_t + \varepsilon \mathbf{v} \cdot \nabla v + u = -P_y + \tau^x F^v(z),$$
$$P_z = -\rho,$$
$$\sqrt{\gamma} \rho_t + \varepsilon \mathbf{v} \cdot \nabla \rho = \gamma w,$$
$$u_x + v_y + w_z = 0.$$  \hspace{1cm} \text{(2.2)}

The surface, bottom, and lateral boundary conditions are:

$$w = 0 \quad \text{on } z = 0,$$
$$w = uB_x + vB_y \quad \text{on } z = -1 + B(x, y),$$
$$u = 0 \quad \text{at } x = 0,$$
$$\frac{\partial w}{\partial x} \to 0 \quad \text{as } x \to -\infty.$$  \hspace{1cm} \text{(2.3)}

We have used notations $u, v, w$ to represent the $x, y, z$ components of vector velocity $\mathbf{v}$, $\tau^x$ is alongshore wind stress, $\rho(x, y, z, t)$ and $P$ are the perturbation density and pressure, respectively. The dimensionless parameters, $\varepsilon$ and $\gamma$, are Rossby and Burger’s numbers as defined below:

$$\varepsilon = \frac{U}{fL}, \quad \gamma = \left(\frac{NH}{fL}\right)^2.$$  \hspace{1cm} \text{(2.4)}
where $U$ is the horizontal velocity scale, $H$ and $L$ are the depth and length scales, $f$ is the Coriolis parameter, and $N$ is a constant representing the buoyancy frequency.

In the momentum equations, we use the vertical shear friction profiles $F^u(z)$ and $F^v(z)$ as body force to parameterize the wind stress; these profiles are defined as

$$
F^u(z) = q_u \left\{ \sin [\pi z + \sin \pi z] + \alpha (1 + z)^{1.5} \right\} e^{\cos \pi z}, \\
F^v(z) = q_v \left\{ \cos [\pi z + \sin \pi z] + e^{-1 \cos \pi z} \right\} e^{\cos \pi z},
$$

where $q_u = -e$, $q_v = e$, and $\alpha = 0.9933$ are chosen to normalize the functions such that

$$
\int_{-1}^{0} F^u(z) \, dz = 0, \quad \int_{-1}^{0} F^v(z) \, dz = 1.
$$

The vertical friction profiles are called Ekman shear friction profiles because of their similarity to an Ekman spiral (see Fig. 4), similar to those used by Song et al. (2001). Physically, atmospheric forcing acts on the ocean surface and penetrates to the ocean body due to the viscosity of water, which is parameterized in numerical oceans. Although the parameterization of subgrid-scale processes is an unsolved problem, its vertical profile is known qualitatively. Here, we use $F^u(z)$ and $F^v(z)$ as a body force to represent the wind stress; their forms can be arbitrary, but the ones we chose simplify the analytical solutions.

Examining the upwelling features in Fig. 1, we observe that the multiple upwelling centers have alongshore spacings of roughly 125 km. Therefore, the characteristic scales for the northern California coast can be chosen as $L \sim 125$ km for horizontal length scale, $H \sim 200$ m for vertical length scale, $\tau^v \sim 1$ dyn/cm$^2 = 10^{-1}$ Pa for wind stress, $f \sim 10^{-4}$ s$^{-1}$ for the Coriolis parameter, and $\overline{\rho}_z \sim 0.25 \times 10^{-2}$ kg/m$^4$ for the mean stratification. Using these values, the buoyancy frequency, and the scales of velocity and time can be calculated:

$$
N^2 = -\frac{g \overline{\rho}_z}{\rho_0} \sim 0.25 \times 10^{-4} \mbox{ s}^{-2}, \\
U = \frac{\tau^v}{\rho_0 H} \sim 0.5 \mbox{ cm/s}, \\
T = \frac{L}{NH} \sim 1.25 \times 10^5 \mbox{ s} \approx 1.5 \mbox{ day},
$$

and the dimensionless Rossby and Burger’s numbers are

$$
\varepsilon \approx \frac{2}{5} \times 10^{-3}, \\
\gamma \approx \frac{3}{5} \times 10^{-2}.
$$

Clearly, these two numbers have the relations, $\varepsilon \ll \gamma \ll 1$. Following Killworth (1978), terms of $O(\varepsilon)$ can be neglected. After linearizing Eqs. (2.2), we will seek approximate expressions for horizontal velocity components $u$ and $v$ in terms of pressure $P$. Operating on the momentum equations with $\sqrt{\gamma} (\partial / \partial t)$ and cross-adding each other gives
As the bracketed terms are an order of magnitude smaller than other terms in the equations, neglecting the bracketed terms yields the relations

\[
\frac{\gamma u_t}{u} = -\sqrt{\gamma} P_{xt} - P_y + \tau^v F^v(z), \quad \frac{\gamma v_t}{v} = -\sqrt{\gamma} P_{yt} + P_x - \tau^v F^w(z). \]

As the bracketed terms are an order of magnitude smaller than other terms in the equations, neglecting the bracketed terms yields the relations

\[
\begin{align*}
[\gamma u_t] + u &= -\sqrt{\gamma} P_{xt} - P_y + \tau^v F^v(z), \\
[\gamma v_t] + v &= -\sqrt{\gamma} P_{yt} + P_x - \tau^v F^w(z).
\end{align*}
\]

Substituting these into the continuity equation results in the pressure equation

\[
\frac{\partial}{\partial t} \left( P_{zz} + \frac{1}{\gamma}(P_{xx} + P_{yy}) \right) = 0,
\]

noticing that $\tau^v$ and $F(z)$ are independent of $x$ and $y$. To seek a solution for our problem, we need to initialize Eq. (2.12). Assuming the second-order derivatives inside the brackets be zero at $t = 0$ gives the following elliptic equation.
\[ P_{zt} = 0 \quad \text{on } z = 0, \]
\[ P_{zt} = -\sqrt{\gamma}(uB_x + vB_y) \quad \text{on } z = -1 + B, \]
\[ \sqrt{\gamma}P_{zt} + P_y = \tau^y F^y(z) \quad \text{at } x = 0, \]
\[ P_{xx}, P_{yy} \to 0 \quad \text{as } x \to -\infty. \]

It should be pointed out that the offshore boundary condition is slightly different from that in Killworth (1978), where \( P_z \to 0 \) is required. To have more choices of initial conditions, we relax the boundary condition to require the second derivatives of the pressure field to approach zero, i.e., \( P_{xx}, P_{yy} \to 0 \) as \( x \to -\infty \). For example, we will seek a solution with initials \( \bar{u} = 0, \bar{v} = \tau^y, \bar{p} = \tau^y(y + x) \), in which the alongshore flow is balanced by the cross-shore pressure gradient and the alongshore pressure gradient is balanced by the constant wind stress. Note the overbar denotes vertical average over the water column. The initials clearly satisfy the above boundary conditions.

2.2. Transformation of coordinates

In order to solve the above problem, we introduce a new east–west coordinate, \( \xi \), as in Killworth (1978), measured from the coastline to an offshore location and scaled by the square root of Burger’s number, \( \sqrt{\gamma} \). Let \( x = \sqrt{\gamma} \xi \), then the elliptic equation (2.13) becomes

\[ P_{zz} + P_{\xi \xi} = O(\gamma). \]  

This assumes that alongshore variations of depth are slow, on the scale of \( L_y = L \), while the offshore variations of depth are on the scale of \( L_x = \sqrt{\gamma}L \). Such a ‘long wave’ assumption has been used in many coastal-trapped-wave literatures (e.g., Allen, 1980; Brink, 1986). This assumption is well justified by the bathymetry shown in Fig. 2, each topographic feature has an alongshore scale of roughly 125 km within a narrow strip of about 40 km in the cross-shore direction. By neglecting the term \( O(\gamma) \), the equation is simplified thus:

\[ P_{zz} + P_{\xi \xi} = 0. \]  

It is convenient to transform Eq. (2.16) into the \( \sigma \)-coordinate system \( z = (1 - B)\sigma \).

Then the equation becomes

\[ P_{\sigma \sigma} + (1 - B(\xi, \gamma))^2 P_{\xi \xi} = 0 \]  

and the corresponding boundary conditions become

\[ P_{\sigma \sigma} = 0 \quad \text{on } \sigma = 0, \]
\[ P_{\sigma \sigma} = P_{\xi}B_{\xi} - P_{\xi}B_{\gamma} \quad \text{on } \sigma = -1, \]
\[ P_{\xi \xi} + P_y = \tau^y F^y(\sigma - B\sigma) - \sigma B_y P_\sigma \quad \text{at } \xi = 0, \]
\[ P_{\xi \xi}, P_{\xi \gamma} \to 0 \quad \text{as } \xi \to -\infty. \]

In the following section, we will solve this problem for pressure and then derive the velocity and density fields from the pressure.
2.3. The perturbation method

Even with the above described simplifications, analytical solutions to Eq. (2.17) subject to (2.18) are very difficult to find. Traditionally, coastal-trapped-wave solutions are solved by eigenvalue method (e.g., Gill and Clarke, 1974; Brink, 1986), in which the coefficient in Eqs. (2.17) and (2.18) have to be constant. In our case, the bottom topography $B(\xi,y)$ in (2.17) varies in both alongshore and cross-shore directions. In addition, the introduction of the forcing functions $F^u$ and $F^v$ also complicates the boundary conditions.

However, with the assumption that the topographic variation, $B(\xi,y)$, is small compared to the total depth, the perturbation method of Killworth (1978) is justified. The perturbation method uses a small parameter in the given equations or boundary conditions, and expands the solution to a series of powers of the parameter. By inserting this expansion into the equations and boundary conditions, a series of simpler equations is obtained. The sum of solutions to the leading equations usually yields a good approximation to the given problem.

We assume $B(\xi,y) = b(\xi,y)\delta$, where $b(\xi,y) = O(1)$ and $\delta$ is the ratio of the topographic variation depth to the total depth, i.e.,

$$\delta = \frac{\Delta h}{H},$$

(2.19)

where $\Delta h$ is the maximum topographic variation and $H$ is the total depth. It should be noted that $\delta$ is the range of the cross-shore slope and the alongshore variation and has to be small relative to 1. For example, $\delta = 0.2$ represents a 20% variation of topographic depth. Because of the ‘long wave’ assumption, the topographic variation corresponds a stronger slope in the cross-shore direction (a scale of $\sqrt{\gamma}L$) than in the alongshore direction (a scale of $L$). Clearly, this is the case for the topography off the central California coast (Fig. 2) and many other coastal oceans as well.

With this assumption, we can expand the pressure in a power series of the topographic variation as

$$P = P_0 + P_1\delta + O(\delta^2),$$

(2.20)

where $P_0$ and $P_1$ are the zero- and first-order approximations of the pressure, respectively. Similarly, we expand the vertical shear stress function in a Taylor series as

$$F(\sigma - B\sigma) = F(\sigma) - \delta b\sigma F_u(\sigma) + O(\delta^2).$$

(2.21)

The method for solving the problem with varying topography in both directions is to separate the variables, one of the widely used means of solving complicated partial differential equations. For simplicity, we assume the bottom topography has the form

$$b(\xi,y) = \beta(y)e^{\xi\xi},$$

(2.22)

in which the topography has an e-fold slope in the cross-shore direction and an arbitrary variation in the alongshore direction. Specifically, a value of $\delta = 0$ represents a flat-bottom case. If $\delta > 0$, a specific topographic variation $\beta(y)$ has to be given. For examples, choosing $\beta(y) = 1 - 0.5\cos(2\pi y)$ gives a sinusoidally alongshore-varying topography and $\beta(y) = 1 - 0.5e^{-8\pi(y-0.5)^2}$ gives a submarine canyon.
3. Model solutions

3.1. General solutions

Inserting the expansions (2.20) and (2.21) into the given problem (2.17) and (2.18), we first obtain the zero-order equation which corresponds to the flat-bottom problem:

$$P_{0,\sigma} + P_{0,\xi \xi} = 0$$  \hspace{1cm} (3.1)

with boundary conditions

$$P_{0,\sigma t} = 0 \quad \text{on} \quad \sigma = 0, -1,$$
$$P_{0,\xi t} + P_{0,\eta} = \tau^\nu F^\nu(\sigma) \quad \text{at} \quad \xi = 0,$$
$$P_{0,\xi \xi}, P_{0,\eta \eta} \to 0 \quad \text{as} \quad \xi \to -\infty,$$

where the subscript 0 indicates the zero-order approximation. Similarly, we can derive the first-order equation

$$P_{1,\sigma} + P_{1,\xi \xi} = 2b(\xi, \eta)P_{0,\xi \xi} - 2\sigma b_\xi P_{0,\xi} - \sigma b_\eta P_{0,\eta}$$  \hspace{1cm} (3.2)

with boundary conditions

$$P_{1,\sigma t} = 0 \quad \text{on} \quad \sigma = 0,$$
$$P_{1,\eta t} = b_\xi P_{0,\eta} - b_\eta P_{0,\xi} \quad \text{on} \quad \sigma = -1,$$
$$P_{1,\xi t} + P_{1,\eta} = -\tau^\nu b_\sigma F^\nu(\sigma) - \sigma \left(b_\xi P_{0,\sigma t} + b_\eta \sigma P_{0,\xi}\right) \quad \text{at} \quad \xi = 0,$$
$$P_{1,\xi \xi}, P_{1,\eta \eta} \to 0 \quad \text{as} \quad \xi \to -\infty.$$

Following Song et al. (2001), we introduce the following three functions

$$\Phi(\xi, \sigma) = \frac{1}{\pi} e^{\pi \xi \cos \pi \sigma} \cos(e^{\pi \xi} \sin \pi \sigma),$$
$$\Psi(\xi, \sigma) = \frac{1}{\pi} e^{\pi \xi \cos \pi \sigma} \sin(e^{\pi \xi} \sin \pi \sigma),$$

$$q(\sigma) = e^{\cos \pi \sigma} \cos(\pi \sigma + \sin \pi \sigma).$$  \hspace{1cm} (3.3)

The flat-bottom solution can be solved in the form

$$P_0 = \tau^\nu q_0 \{y + \sqrt{y} \xi\} + \tau^\nu \Phi(\xi, \sigma).$$  \hspace{1cm} (3.4)

Notice that $q_0 = e^{-1}$ and $q_\nu = -q_u = e$ is factorized into $\tau^\nu$ for simplicity. Using the relations in (2.11), the flat-bottom solution can be written as

$$u_0 = \tau^\nu q(\sigma) - \tau^\nu \Phi_\xi,$$  \hspace{1cm} (3.5)
$$v_0 = \frac{\tau^\nu}{\sqrt{y}} \Phi_\xi - \tau^\nu \{F^\nu(\sigma) - q_0\},$$  \hspace{1cm} (3.6)
$$w_0 = \frac{\tau^\nu}{\sqrt{y}} \Psi_\xi,$$  \hspace{1cm} (3.7)
$$\rho_0 = \tilde{\rho}(x) + \tau^\nu \Psi_\xi.$$  \hspace{1cm} (3.8)
The first-order equation is solved in Appendix A and has the form

$$P_1 = \frac{t^2}{2} \varepsilon^y \beta F_2(\xi, \sigma) + t \varepsilon^y \beta F_1(\xi, \sigma) + \varepsilon^y Q(\xi, \sigma; y, t),$$

(3.11)

where $F_1$, $F_2$, and $Q$ are known functions defined in Appendix A. Once the pressure is known, the first-order topographic perturbation velocity fields and density can be derived as

$$u_1 = -\frac{t^2}{2} \varepsilon^y \beta_{yy} F_2(\xi, \sigma) - t \varepsilon^y \beta_y \{F_2 + F_1 - \sigma \Psi \varepsilon^y (\xi, \sigma) e^{n \xi}\} - \varepsilon^y \beta \{[\sigma D_2 + \xi D_1] e^{n \xi}\}$$

$$+ G(\xi, y) + \sigma [\Psi \varepsilon^y (\xi, \sigma) - F(\sigma)] e^{n \xi},$$

(3.12)

$$v_1 = -\varepsilon^y \sqrt{SF_2(\xi, \sigma) + \beta_y [F_1 - \sigma \Psi \varepsilon^y e^{n \xi}] + Q_y} + \frac{\varepsilon^y}{\sqrt{F}} \left\{ \frac{t^2}{2} \beta_y F_2 \xi + t \beta [F_1 \xi - \sigma \Psi \varepsilon^y e^{n \xi}] + Q_\sigma \right\}$$

$$- \varepsilon^y \beta \sigma F_{yy} e^{n \xi},$$

(3.13)

$$w_1 = -\varepsilon^y \sqrt{F} \{t \beta_y F_2 \sigma + \beta [F_1 \sigma - \Psi \varepsilon^y e^{n \xi}] + Q_{st}\},$$

(3.14)

$$\rho_1 = -\left\{ \frac{t^2}{2} \varepsilon^y \beta_{tt} F_2(\xi, \sigma) + t \varepsilon^y \beta [F_1 \xi - \sigma \Psi \varepsilon^y e^{n \xi}] + \varepsilon^y Q_\sigma \right\},$$

(3.15)

where $Q_\xi$, $Q_{st}$, $Q_\sigma$, and $Q_{at}$ are derivatives of $Q(\xi, \sigma; y, t)$. Their analytical forms are

$$Q_\xi = \sum_{n=1}^{\infty} q_n n \pi \int_0^{y-t/n\pi} \beta(\eta) d\eta e^{n \xi} \cos n \pi \sigma,$$

$$Q_{st} = -\sum_{n=1}^{\infty} q_n \frac{1}{n \pi} \beta_y (\eta - \frac{t}{n \pi}) e^{n \xi} \cos n \pi \sigma,$$

$$Q_\sigma = \sum_{n=1}^{\infty} q_n n \pi \int_0^{y-t/n\pi} \beta(\eta) d\eta e^{n \xi} \sin n \pi \sigma,$$

$$Q_{at} = -\sum_{n=1}^{\infty} q_n \beta (\eta - \frac{t}{n \pi}) e^{n \xi} \sin n \pi \sigma.$$

Combining the zero-order solutions in (3.7)–(3.10) and the first-order solutions in (3.12)–(3.15), results in the full solutions to problem (2.1)–(2.6)

$$u = u_0 + u_1 \delta + O(\delta^2),$$

(3.16)

where $\delta$ is the perturbation scale, i.e., the ratio of the topographic variation to the total depth. In the present study, we chose $\delta = 0.2$. In this case the third terms in the above solutions are about two magnitudes smaller than the leading terms and can be neglected. The same form of the full solution can be given for variables $v$, $w$, and $\rho$.

The terms with $Q_{st}$, $Q_\xi$, and $Q_{at}$ are the components of topographic trapped Kelvin waves; these waves are boundary phenomenon in the sense that the amplitude is large near the coast and decays away from it, because of the factor $e^{n \xi}$. In addition, the waves propagate forward with
boundary on the right (to the north on the West Coast as \( y - (t/n\pi) = c \)) no matter whether the winds blow southerly or northerly. These waves are internal Kelvin waves with internal radius of deformation \( \lambda_n = \sqrt{\gamma / np} \), where \( \sqrt{\gamma} = NH/fL \).

3.2. Solution with a uniform slope

In the case without alongshore topography, i.e., \( \beta(y) = 1 \), the bottom topography is derived from \( b(\xi, y) = e^{\alpha\xi} \), which corresponds to a coastal region with a uniform self/slope. The cross-section of the solution at \( t = 6 \) is shown in Fig. 5. A uniform upwelling front is well-developed at this time and the wind-induced Ekman transport in the surface layer (dashed lines in Fig. 5a) is away from the coast, and an onshore movement of water (solid lines in Fig. 5a) occurs at depth below the surface. The alongshore flow in Fig. 5b shows a surface coastal jet following the wind direction to the south and the returning undercurrent below it. The undercurrent is much weaker than the surface jet. Fig. 5c shows that the upwelling occurs near the coast to preserve continuity.

![Theoretical Solution](image)

Fig. 5. Cross-section of the solution with a uniform slope topography at \( t = 6 \): (a) cross-shore velocity \( u \), solid and dashed lines represent onshore and offshore flow, respectively; (b) alongshore velocity \( \sqrt{v} \), solid and dashed lines represent north and south flow; (c) vertical velocity \( \sqrt{w} \); (d) density \( \rho \). Domain dimensions are \( L_x = 0.3L \) and \( L_z = H \). Units are nondimensional.
The isopycnals slope upward toward the coast in Fig. 5d, indicating transport of cold bottom water to the surface. The overall structure of the solution in the simple model is qualitatively consistent with known upwelling features (Brink, 1983).

The mechanism generating the undercurrent is explained by McCreary (1981). He suggests that the undercurrent is induced by an alongshore pressure gradient established by the surface forcing of wind stress, through the radiation of Kelvin and Rossby waves. The presence of the alongshore pressure gradient is confirmed by our solution in the pressure equation (3.6). The first term, $\tau^y q_0 y$, represents a poleward pressure gradient force $-p_y = -\tau^y q_0 > 0$. Notice that $\tau^y$ is negative (equatorward wind) and $q_0 = e^{-1}$. A similar term is also given by Killworth (1978). Such an alongshore pressure gradient has been verified by observed sea surface height data. For example, Chelton (1980) presents monthly mean dynamic height of the sea surface along the west coast of North America. His data show that sea level drops roughly 16 dyn cm from 25° N to 38° N in the autumn and 8 dyn cm in the spring. In our model the coastal region from 25° N to 38° N represents a length scale of $\eta = 13$. Noticing $\tau^y$ is scaled by $e$, we obtain that our solution has a sea level drop of about 13 dyn cm, which is in the range of the observations.

3.3. Solution with sinusoidal-varying topography

In the case with alongshore topography, we choose $\beta(y) = 1 - 0.5 \cos(2\pi y)$ to represent the bathymetry, which yields bottom topography of $b(\xi, y) = \beta(y)e^{\eta \xi}$, where $0 \leq y \leq 3L$, $-L \leq \xi \leq 0$ or $-\sqrt{\eta}L \leq x \leq 0$, and $L = 1$ in nondimensional units. This bathymetry (solid lines in Fig. 6) represents the three topographic highs in the alongshore direction. The model surface and bottom temperature and velocity fields are shown in Fig. 6. With the alongshore-varying topography, the solution shows varying upwelling fronts, meandering surface (left panel) and bottom (right panel) currents. Three upwelling centers are found on the downstream sides (referred to the direction of upwelling favorable winds, i.e., $\tau^y h_0 > 0$) of these topographic highs and each corresponds a cyclonic circulation with a tongue of cold water projecting in the offshore direction and interleaving with a tongue of warmer water towards the coast. Particularly, the surface current (left panel) has a feature of turning offshore (anticyclonically) when encounters a topographic high, and turning back shoreward (cyclonically) when passes the topographic high. These meandering features are consistent with those observed in the laboratory experiments of Narimousa and Maxworthy (1986). They reported that topography (a ridge) could produce standing offshore intrusion or waves at the ridge and downstream, which transported upwelled water far from the mean position of the front.

To further examine the alongshore-varying structure of the solution, we also plot an alongshore section of vertical velocity and density in Fig. 7. Both variables show enhanced upwelling on the downstream sides of the topographic highs and weakened upwelling on the upstream sides. The density fields show that the surface water temperature is colder on the southern sides than on the northern sides. Such upwelling features have often been reported near Cape Mendocino (Magnell et al., 1990) and Cape Blanco (Barth et al., 2000).
Several dynamical mechanisms have been suggested to explain why topographic variations may cause alongshore-varying fronts and meandering currents (e.g., Arthur, 1965; Narimousa and Maxworthy, 1986; Mitsudera and Grimshaw, 1991). Killworth (1978) suggests that it is the bottom boundary condition that leads to upwelling and downwelling distributions. This can be verified by our solution of vertical velocity at the bottom layer:

\[
wb = -uh_x - vh_y = -\frac{\tau^v}{\sqrt{\gamma}} \left\{ b_y q_0 + t b_y e^{\pi^2 t} e^{-\gamma t} \right\}.
\] (4.1)

The first term in the brackets is the effect of the slope on the bottom vertical velocity. It is positive because \(q_0\) and \(- (\tau^v/\sqrt{\gamma})\) are positive numbers. The second term would become dominant when \(t\) increases and its sign is determined by \(b_y\). Therefore, there will be downwelling induced on the upstream side \((b_y < 0)\) and upwelling induced on the downstream side \((b_y > 0)\), referring to the wind direction, of a topographic high.

In fact the topographic effect is at work as soon as the wind forcing is applied; it becomes intensive with time, and causes downstream upwelling and upstream downwelling distributions.
The effect on the meandering currents can be revealed by rewriting Eq. (2.12) into the perturbation form of the vorticity equation,

$$\frac{\partial}{\partial t} \{ -\rho_z + \gamma \zeta \} = 0,$$

where $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$ is the relative vorticity. Upwelling leads to upward isopycnals to the coast and downwelling is induced on the other side (upstream). Wind stress is uniform. Domain dimensions are $$L_y = 3L$$ and $$L_z = H$$. Units are nondimensional.

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Fig. 7. An alongshore section of the solution near the coast (or 95% $$L_x$$ from the offshore boundary): (a) vertical velocity $$\sqrt{\gamma w}$$ and (b) density. Notice that upwelling is enhanced on the southern side (downstream) of a topographic high and downwelling is induced on the other side (upstream). Wind stress is uniform. Domain dimensions are $$L_y = 3L$$ and $$L_z = H$$. Units are nondimensional.
laboratory experiments of Narimousa and Maxworthy (1986), in which the offshore intrusion of cyclonic circulations is referred as downstream standing wave. Such a process only occurs in stratified flows as the relationship of
\[ q_t = \sqrt{f \Delta \rho_w}, \]
which suggests that the combined effect of topography and stratification is responsible for the alongshore variations of upwelling fronts and currents.

4.1. Sensitivity analysis

As we hypothesized that the combined effect of topography and stratification is responsible for the upwelling front variations, we like to explore the sensitivity of the solution to their physical parameters. They are the ratio of the topographic variation depth to the total depth, \( \delta \), and Burger’s number of stratification, \( \gamma \). We first define an alongshore variation index,
\[ d_{\rho} = \max \{ \Delta \rho_{\max}(x) \}, \]
where \( \Delta \rho_{\max} \) is the maximum variation of density at an alongshore section, \( x \). Clearly, the index is a number measuring the maximum density variation in the alongshore direction.

To calculate \( d_{\rho} \), we can insert density (3.15) into Eq. (4.2) because Eq. (3.10) has no alongshore variation. Taking \( \beta(y) = 1 - 0.5 \cos(2\pi y) \) as an example, we have \( \Delta(\beta_{\gamma})_{\max} = 2\pi, \Delta(\beta)_{\max} = 1 \), and \( \Delta(Q_{\sigma})_{\max} = tq_0 \). Neglecting the higher order term \( \mathcal{O}(\delta^2) \), we have
\[ d_{\rho} = \gamma \delta (\hat{f}t' \pi \tau) \max(F_{2\sigma}) + \sqrt{\gamma} \delta \hat{f}t' \tau \max(F_{1\sigma} - \Psi_{\chi} e^{\pi \hat{z}} + q_0), \]
where \( t' = t/\sqrt{\gamma f} \) is the dimensional time. Our calculations show that the first term is an order of magnitude larger than the second one when \( \hat{f}t' > 1 \). Therefore, neglecting the second term, we have a bilinear function:
\[ d_{\rho} = C(t') \gamma \delta, \]
where \( C(t') = (\hat{f}t')^2 \pi \tau \max(F_{2\sigma}) \) is the function of time \( t' \) and it can be estimated as about 160 when \( t = 6 \) days. Eq. (4.4) indicates that the alongshore variation is dependent upon a bilinear function of two important parameters—the ratio of the topographic variation depth to the total depth and Burger’s number of stratification. The larger the product of the two parameters is, the stronger the variation is.

4.2. Cross-shore slope

As we mentioned in the introduction, topographic variations in both alongshore and cross-shore directions can be important. Here we focus on the cross-shore direction. By comparing the solution (Fig. 6) with those of Killworth (1978) and Song et al. (2001), we notice that the new solution with a cross-shore slope has a continuous undercurrent, mainly following topographic contours (right panel). The continuity of the poleward undercurrent along the eastern boundary of the mid-latitude north Pacific has been observed and discussed by Pierce et al. (2000). However, the poleward solutions of Killworth (1978) and Song et al. (2001) do not have this continuity. In addition, the surface current (left panel) turns back shoreward (cyclonically) when passes a
topographic high, which is not observed in the solutions of Song et al. (2001). Such a cyclonic circulation with a tongue of cold water on its inshore side has been suggested by Barth et al. (2000) as a result of stretching and deepening process. It indicates that the cross-shore slope plays a role in maintaining the meandering structure of the alongshore currents. The anticyclonic circulation can further induce downwelling on the offshore side of the current, while the cyclonic circulation enhances upwelling and forms an upwelling center on the inshore side of the current.

Pedlosky (1978) also considered the slope effect by comparing solutions obtained with and without a topographic slope in his two-dimensional cross-shore model. He concluded that the effect of topographic slope was to reduce the alongshore velocity induced by upwelling, as a consequence of the reduced slope of the isopycnal surfaces required to bring them to the surface. Our model produces somewhat different results. The alongshore velocity profile at $x = -0.1$ (i.e., 10% $L_x$ from the coast). Units are nondimensional.

![Comparison of solutions between flat-bottom case (solid line) and slope-bottom case (dashed line). Plots are the alongshore velocity profiles at an offshore location $x = -0.1$ (i.e., 10% $L_x$ from the coast). Units are nondimensional.](image)

Fig. 8. Comparison of solutions between flat-bottom case (solid line) and slope-bottom case (dashed line). Plots are the alongshore velocity profiles at an offshore location $x = -0.1$ (i.e., 10% $L_x$ from the coast). Units are nondimensional.

It should be noted that, in Pedlosky's model, the isopycnals are assumed to be initially parallel to the bottom slope, and the alongshore velocity is mainly balanced by the cross-shore pressure gradient force as the delta function sink brings the isopycnals to the surface. As the delta function sink at the coast is infinitely large, the slope increase of isopycnals to the surface in the flat-bottom case is greater than in the slope case; therefore, alongshore velocity is stronger in his flat-bottom case. In our model, the isopycnals are assumed to be initially flat whether there is a slope or not; they can also intersect with the slope. However, kinematic bottom boundary conditions (2.3) cause a slope effect on the vertical velocity and therefore enhance the alongshore velocity.
5. Cross-shore exchange

In this section we will focus on the consequences of the alongshore variations on cross-shore exchange of water mass. Song et al. (2001) provides an upwelling conservation theorem that the topographic variation does not change the total amount of upwelled water. We like to verify it with the new solution. The total \( y \)-integrated upwelling (or total Ekman pumping) can be written as

\[
M_v = \int_0^L w_1 \, dy = -\frac{\tau^y}{\sqrt{\gamma}} \beta L \left\{ F_{1\sigma} - \Phi_{1\sigma} e^{\gamma \xi} - \sum_{n=1}^{\infty} q_n e^{i n \xi} \sin n \pi \sigma \right\},
\]

and its change with respect to time is

\[
\frac{\partial}{\partial t} M_v = 0,
\]

where \( \beta \) is the alongshore mean topography. This suggests that the total upwelled water (Ekman pumping) is conserved, independent of time, and equal to the uniform slope case.

The implication of the conservation theorem is that the redistribution of upwelling from one side of a topographic high to the other side of the topographic high results in enhanced cross-shore exchange of water. This mechanism can be verified by calculating the cross-shore overturning function

\[
\Gamma(\xi, z) = \frac{1}{l_2 - l_1} \int_z^0 \int_{l_1}^{l_2} u(\xi, y, z) \, dy,
\]

which should give a constant value over a full periodic segment \([0, L]\), but increases or decreases with time over a half of the segment. The results are given in Fig. 9, where the left panels are the total overturning for the full periodic segment of \([0, L]\) and the right panels are for the half segment of \([0, L/2]\), respectively. It can be seen that the total cross-shore overturning ((a) at \( t = 1 \) and (c) at \( t = 8 \)) does not change with time, but the local cross-shore overturning ((b) at \( t = 1 \) and (d) at \( t = 8 \)) decreases with time due to the local topographic effect.

The topographic effect on upwelling fronts may seem obvious, but quantifying how strong the effect is on cross-shore exchange may not be quite straightforward. In order to examine the effect of topography on the cross-shore exchange of water masses, we calculated the vertically averaged cross-shore velocity separately

\[
q^+ = \frac{1}{h} \int_{-h}^0 u^+ \, dz, \quad q^- = \frac{1}{h} \int_{-h}^0 u^- \, dz,
\]

where \( u^+ = \max\{0, u\} \), \( u^- = \min\{0, u\} \), \( q^+ \) represents onshore transport (mostly bottom cold water), and \( q^- \) represents offshore transport (mostly surface warm water). Fig. 10 gives the results for the total vertically averaged offshore velocity (left panel), showing concentrated offshore transports near the topographic highs, and for the surface layer-averaged onshore velocity (right panel), showing concentrated onshore transports near the topographic lows. These results suggest that cross-shore exchange of water masses is greatly enhanced by topographic features. This corresponds to the tongues of cold upwelling water observed in coastal oceans (Brink and Cowles,
that project in an offshore direction and interleave with tongues of warmer water moving towards the coast resulting in meandering fronts and eddies. Particularly, the intense meandering surface currents transport upwelled water far from the mean position of the front.

Fig. 9. Cross-shore overturning stream function calculated from the solution with alongshore-varying topography: left panels (a) and (c) for the total segment of \([0, L]\) at time \(t = 1\) and \(t = 8\), respectively; right panels (b) and (d) for the half segment of \([0, L/2]\) at time \(t = 1\) and \(t = 8\), respectively. The downwelling cell at depth corresponds to the depressed isopycnals. Units are nondimensional.
6. Summary and discussions

In this paper we have extended the theoretical three-dimensional wind-driven coastal ocean model of Killworth (1978) to allow topographic variations in both alongshore and cross-shore directions. The vertical shear friction profiles of Song et al. (2001), similar to those in Ekman’s solutions, are used to represent Ekman dynamics for the momentum equations. The role of bottom topography in coastal upwelling dynamics and the cross-shore exchange of water masses is examined in detail.

In the case without alongshore-varying topography, we obtain a simple two-dimensional solution of wind-driven upwelling circulation. This solution reveals the classical features of coastal upwelling circulations (Pedlosky, 1978) and qualitatively agree with observations (Brink, 1983). In the case with topographic variations in both alongshore and cross-shore directions, the solution is found to form meandering currents and multiple upwelling centers along the coast. Each of the upwelling centers, associated with a cyclonic circulation of surface current, is located on the downstream side of a topographic high, similar to the features observed by the satellite imagery (Fig. 1) and the laboratory experiments of Narimousa and Maxworthy (1986).
Analyses of the solutions lead us to propose the following hypotheses. (1) The variation of upwelling fronts and currents is believed to be caused by the combined effect of topography and stratification. Topographic variation causes uneven upwelling distribution and leads to density variation, which results in a varying horizontal pressure gradient field that causes the meandering currents. The variation index is dependent upon a bilinear function of two physical parameters—the ratio of the topographic variation depth to the total depth $\delta$ and Burger’s number of stratification $\gamma$. (2) Cross-shore slope is found to play a role in maintaining the meandering structure of the alongshore currents. The anticyclonic circulations can further induce downwelling on the offshore side of the current, while the cyclonic circulations enhance upwelling and form upwelling centers on the inshore side of the current. (3) Alongshore topography does not change the total upwelled water, i.e., total Ekman pumping (cross-shore overturning) is conserved. It causes variation of upwelling fronts by inducing downwelling upstream ($\tau^y h_x < 0$) and enhancing upwelling downstream ($\tau^y h_x > 0$) of a topographic high. Consequently, it increases the cross-shore exchange of water.

It should be pointed out that there are several limitations in the proposed theory. The model is less effective on nearshore flows and deep slope regions. This limitation is primarily due to the mathematical difficulty to relax the ‘long wave’ simplification. Observational studies (Lentz and Winant, 1986; Lee et al., 1989; Lentz et al., 1999) reveal dynamically distinct regions and momentum balances: the surf zone (water depth of order 1 m), the inner-shelf (10 m), and the mid-shelf (30 m). In the surface zone the alongshore momentum balance is between the cross-shore gradient of the wave radiation stress and the bottom stress. On the inner-shelf the alongshore momentum balance is mainly frictional after a few days of wind forcing; the alongshore wind stress and pressure gradient are balanced by bottom stress. There is no undercurrent observed at depth. At mid-shelf the alongshore momentum balance is less frictional and hence flow accelerations are important. The cross-shore momentum balance is predominantly geostrophic. Our model fails to describe the nearshore dynamics, at least in the surf zone and inner-shelf regions because both friction and bottom stress are ignored. It is not known where a bottom boundary layer can be included in the model by reversing the proposed Ekman layer profiles. Analytical solutions to this problem have yet to be found: the challenge is to include the shallow region where the surface and bottom layers meet.

The limitation of time scales and the lack of nonlinear dynamics in the model also make the theory inapplicable to the full cycle of the upwelling circulation. Killworth (1978) provides a time scale of 26 days for the neglected nonlinear terms to become an effect on the model solution. To estimate the nonlinear effects on our solutions, we directly calculated the velocity tendency and nonlinear terms using the model solutions. The results are given in Fig. 11, indicating that the nonlinear terms may not be negligible after 12 days of wind forcing, which are shorter than the estimate of Killworth’s scaling analysis. Nevertheless, the model only explains the initial development of the meandering currents and fronts and their relationship with the local topography qualitatively. In addition, the established meandering currents and fronts could become unstable (Ikeda and Emery, 1984; Barth, 1989), develop filaments (Strub et al., 1991), and separate offshore jets (Barth et al., 2000), which, in one way or another, may be related to the underlying topography. The full cycle of the upwelling circulation could be far more complex than our analytical solutions, which may be difficult to represent with a theoretical model. We are currently studying this problem with numerical models and results will be reported in a future publication.
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Appendix A. First-order solution

Here we will solve the first-order perturbation problem (3.3) and (3.4) with the topography given by (2.22). The desired solution is assumed to have the form
\[ P_1 = -\sigma b(\zeta, y) P_0 + B_0(y, t) + \sum_{n=1}^{\infty} \left\{ A_n(y, t)(\sigma \sin n\pi\sigma + \zeta \cos n\pi\sigma) + B_n(y, t) \cos n\pi\sigma \right\} e^{-n\pi\zeta}, \]  
(A.1)

with coefficients \( A_n \) and \( B_n \) determined by the boundary conditions given in (3.4). First, it can be seen that the solution satisfies both the surface condition \( P_1 r = 0 \) (on \( r = 0 \)) and the offshore condition (as \( n! \to \infty \)). Coefficients \( A_n \) and \( B_n \) are then solved to satisfy the remaining bottom and coastal wall boundary conditions. Substituting the solution into the bottom boundary condition (on \( r = \frac{1}{C_0} \)) yields

\[ \sum_{n=1}^{\infty} \left\{ n\pi A_n \right\} (-1)^n e^{n\pi\zeta} = \frac{\tau^y b_{y}}{\sqrt{2}} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)!} e^{n\pi\zeta} - \tau^y b \sum_{n=1}^{\infty} \frac{(-1)^n n\pi}{(n-1)!} e^{n\pi\zeta} - \tau^y b q_0. \]  
(A.2)

Using the assumption (2.22), we split (A.1) into two parts:

\[ \sum_{n=1}^{\infty} C_n (-1)^n n\pi e^{n\pi\zeta} = -\tau^y \beta_{y} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} e^{n\pi\zeta}, \]
\[ \sum_{n=1}^{\infty} D_n (-1)^n n\pi e^{n\pi\zeta} = \tau^y \beta \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!} (n-1)\pi e^{n\pi\zeta} - \tau^y \beta q_0 e^{n\pi\zeta}, \]

where

\[ A_n = \frac{\tau^2}{2} C_n + tD_n. \]

As \( e^{n\pi\zeta} \) are varying functions, we obtain

\[ A_n = \frac{\tau^2}{2} \tau^y \beta_{y}(y) c_n + t \tau^y \beta(y) d_n, \quad n \geq 1, \]  
(A.3)

where

\[ c_n = \frac{-n-1}{\pi n!}, \]
\[ d_n = \begin{cases} \frac{q_0}{(n-1)!} & \text{if } n = 1, \\ \frac{q_0}{n!} & \text{if } n \geq 2. \end{cases} \]

Next, we determine \( B_n \) by requiring the solution (A.1) to satisfy the coastal boundary condition at \( \zeta = 0 \). This yields

\[ \sum_{n=1}^{\infty} (A_{ny} + n\pi A_n) \sigma \sin n\pi\sigma + \sum_{n=1}^{\infty} (B_{ny} + n\pi B_n + A_n) \cos n\pi\sigma = -B_{0y}. \]  
(A.4)

Integrating the above equation from \( \sigma = -1 \) to 0, gives

\[ B_{0y} = \sum_{n=1}^{\infty} (n\pi A_n + A_{ny}) \int_{-1}^{0} \sigma \sin n\pi\sigma d\sigma. \]
From (A.3), we have
\[ n\pi A_{nt} + A_{ny} = \frac{t^2}{2} \tau^y \beta_n c_n + \tau^y \beta [d_n + n\pi c_n] + \tau^y \beta n\pi d_n. \]

Substituting into the above equation, we obtain the first coefficient as
\[ B_0(y, t) = \frac{t^2}{2} \tau^y \beta_n (y) r_0 + t \tau^y \beta (y) e_0 + \tau^y \int_0^y \beta (\eta) \, d\eta g_0, \]
(A.5)
where we have defined
\[ r_0 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} c_n, \quad e_0 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} [d_n + n\pi c_n], \quad g_0 = \sum_{n=1}^{\infty} (-1)^n d_n. \]

Applying the Fourier theorem to (A.3), we obtain the following set of hyperbolic equations:
\[ B_{ny} + n\pi B_{nt} = -A_{nt} - \sum_{m=1}^{\infty} (A_{my} + m\pi A_{mt}) a_{mn}. \]
(A.6)

By introducing coefficients
\[ r_n = -\sum_{m=1}^{\infty} c_m a_{mn}, \]
\[ e_n = -c_n - \sum_{m=1}^{\infty} [d_m + m\pi c_m] a_{mn}, \]
\[ g_n = -d_n - \sum_{m=1}^{\infty} m\pi d_m a_{mn}, \]
we further simplify (A.6) as
\[ B_{ny} + n\pi B_{nt} = \frac{t^2}{2} \tau^y \beta_y r_n + t \tau^y \beta_n e_n + \tau^y \beta g_n \]
(A.7)
with initial condition \( B_n(y, 0) = 0 \).

This equation has the unique solution
\[ B_n(y, t) = \frac{t^2}{2} \tau^y \beta_y (y) r_n + t \tau^y \beta (e_n - n\pi r_n) + \tau^y q_n \int_y^{y-(t/n\pi)} \beta (\eta) \, d\eta, \]
(A.8)
where
\[ q_n = (e_n - n\pi r_n) n\pi - g_n, \quad n \geq 1. \]

In order to derive the solution explicitly, we introduce the following functions:
\[ F_1(\xi, \sigma) = E(\xi, \sigma) - R_\xi(\xi, \sigma) + \sigma D_2(\xi, \sigma) + \xi D_1(\xi, \sigma) + \sigma \psi_\xi e^{\xi}, \]
(A.9)
\[ F_2(\xi, \sigma) = R(\xi, \sigma) + \sigma C_2(\xi, \sigma) + \xi C_1(\xi, \sigma), \]
(A.10)
where $\Phi$ and $\Psi$ are defined in (3.5), and the others are defined as

\[
R(\zeta, \sigma) = \sum_{n=0}^{\infty} r_n e^{n\pi \zeta} \cos n\pi \sigma, \quad (A.11)
\]

\[
E(\zeta, \sigma) = \sum_{n=0}^{\infty} e_n e^{n\pi \zeta} \cos n\pi \sigma, \quad (A.12)
\]

\[
G(\zeta, \sigma) = \sum_{n=0}^{\infty} g_n e^{n\pi \zeta} \cos n\pi \sigma, \quad (A.13)
\]

\[
C_1(\zeta, \sigma) = \sum_{n=1}^{\infty} c_n e^{n\pi \zeta} \cos n\pi \sigma, \quad (A.14)
\]

\[
C_2(\zeta, \sigma) = \sum_{n=1}^{\infty} c_n e^{n\pi \zeta} \sin n\pi \sigma, \quad (A.15)
\]

\[
D_1(\zeta, \sigma) = \sum_{n=0}^{\infty} d_n e^{n\pi \zeta} \cos n\pi \sigma, \quad (A.16)
\]

\[
D_2(\zeta, \sigma) = \sum_{n=0}^{\infty} d_n e^{n\pi \zeta} \sin n\pi \sigma, \quad (A.17)
\]

\[
Q(\zeta, \sigma; y, t) = \sum_{n=1}^{\infty} q_n \int_{-\pi}^{\pi} \beta(\eta) \, d\eta e^{n\pi \zeta} \cos n\pi \sigma + g_0 \int_{0}^{y} \beta(\eta) \, d\eta. \quad (A.18)
\]

Finally, we derive the topography perturbation solution as

\[
P_1 = \frac{\rho^2}{2} \tau^{\nu} b \mathcal{J}_1^2 (\zeta, \sigma) + \tau^{\nu} b \mathcal{J}_1 (\zeta, \sigma) + \tau^{\nu} Q(\zeta, \sigma; y, t). \quad (A.19)
\]

References


