A Mechanism Governing the Estuarine Circulation in Deep, Strongly Stratified Fjords

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The estuarine circulation in deep, strongly stratified fjords is discussed. It is argued that a two-layer description of the stratification in the fjord is correct in very wide fjords (wide compared to the width of the mouth) and possibly also in narrow fjords with high runoff and/or weak mixing. A theory for the thickness and the salinity of the brackish layer in a wide fjord was earlier developed by the present author (and published in Swedish, 1975) and it is presented here.

Among other findings in that theory we may mention that the thickness of the brackish layer is primarily determined by the internal hydraulic control for high specific runoff (runoff/unit horizontal surface area of the fjord) and weak mixing. In the other extreme, with low specific runoff and strong mixing, the thickness of the brackish layer is found to be proportional to the Monin–Obukhov length.

The circulation in not very wide fjords is also discussed. It is found that recirculation in the brackish layer may be expected whereby a two-layer description of the flow field breaks down at least locally. A critical Rayleigh number that seems to control the recirculation of brackish water within the fjord is found. A theory for the density difference between the mouth and the head in the brackish layer in the fjord is developed from the critical Rayleigh number condition. It is found that this density difference normally is proportional to the density difference between the brackish water at the mouth and the underlying sea water provided that the river runoff is not too heavy and/or the mixing is not too weak. This prediction is confirmed by extensive measurements from the Nordfjord.

Introduction

There are in general two main vertical circulation systems in fjords. One in the upper part of the fjord is mainly caused by the freshwater runoff from rivers. The lighter freshwater is mixed with the underlying seawater by winds and tides and a brackish top layer develops. Beneath the brackish layer, seawater flows inwards to an extent determined by the entrainment or mixing of seawater into the brackish water. This circulation system is often termed estuarine circulation. The other circulation system concerns the deep water, the water below the sill level in the fjord. The deep water, being trapped by the sill, can escape from the fjord when lifted above sill level. Vertical mixing processes are usually indispensable for the lifting
and circulation of the deep water. In most fjords these two circulation systems exist together. The interaction between the circulation systems may often be considered as weak and the seawater entrained into the brackish top layer is then taken from an intermediate layer above sill level. In some fjords with narrow connections to the sea (constrictions) the two circulation systems may merge. This is the case in so called overmixed estuaries, a concept introduced by Stommel & Farmer (1953) and explained below, where the deep water is used for dilution of the brackish top layer. The water between the brackish top layer and the deep water is termed the intermediate water. Some basic concepts utilized in fjord dynamics are shown in Figure 1. In this paper we will only discuss the estuarine circulation.

Much attention has in the past been given to the modelling of circulation systems in fjords. A great step forward was taken by Stommel & Farmer (1953) who introduced the concept of hydraulic control in a two-layer system. The (internal) hydraulic control is established in the narrow mouth of the fjord and the critical condition derived by them is

\[ F_{1t}^2 + F_{2t}^2 = 1 \] (at the mouth of the fjord), \( i = 1, 2 \) in the upper and lower layer respectively.

where

\[ F_{it}^2 = \frac{u_i^2}{g'H_i} \]

\( u_i \) and \( H_i \) are velocity and depth respectively of the layer and \( g' = g\Delta \rho/\rho_o \) where \( \Delta \rho \) is the density difference between the two layers, \( \rho_o \) is the density of some typical water in the system and \( g \) is the acceleration of gravity. Equation (1) is valid for stationary currents. If tides or other non-stationary currents are present at the mouth one has to investigate their influence upon equation (1). This topic was to some extent discussed by the present author (Stigebrandt, 1977).

Through combination of equation (1) with conservation equations for salt and mass, taking the freshwater runoff properly into account, Stommel & Farmer (1953) derived an algebraic equation for the salinity in the upper layer of the estuary. From this theory they were able to show that there is an upper limit for the salinity of the brackish water in the estuary. This maximum value of the salinity varies from estuary to estuary. It depends essentially on the salinity in the sea outside the constriction, the freshwater runoff to the
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The maximum salinity can never be exceeded no matter how great a supply of mixing energy the fjord is given. This is the case of overmixing mentioned above.

The physical explanation for overmixing is found in the constriction. There is an upper limit for the two-layer transport in the constriction as explained by, e.g., Stigebrandt (1975). When this is reached the transport capacity of the constriction with respect to two-layer flow is fully utilized. In the referred paper there is a detailed discussion of the solutions of the Stommel & Farmer equation. Some of those solutions are shown in Figure 2 taken from the

Figure 2. The solution to the Stommel & Farmer equation for different values of the estuarine Froude number, \( F_r \). The parameter spaces for N-fjords and O-fjords are indicated.

same paper. On the ordinate is the relative depth, \( \eta = H_2/(H_2 + H_1) \), of the lower layer in the constriction. On the abscissa is the mixing parameter \( P \), which is defined as the ratio between the fluxes of outflowing brackish water and river water. Each curve is a solution to the equation with different but constant values of a parameter \( F_e \), termed the estuarine Froude number. \[ F_e = \frac{Q_f}{(g\Delta \rho_0 / \rho_s)^{1/2} \beta m H_m^{3/2}} \]

where \( Q_f \) is the volume flux of freshwater to the fjord, \( \Delta \rho_0 \) is the density difference between fresh and seawater, \( \beta m \) and \( H_m \) are width and depth respectively of the control section (the fjord mouth). Thus for a constant freshwater runoff to a fjord the \( F_e \) value is constant. From Figure 2 it can be seen that the relative depth of the upper layer in the mouth varies with the mixing parameter \( P \). The curves shown in that figure have a maximal \( P \) value. When an estuary reaches this \( P \) value it is overmixed. The theory is valid for steady exchange in a short constriction. For long constrictions friction will be important and the maximal \( P \) value from the theory above will not be reached (see Assaf & Hecht, 1974). Friction decreases the transport capacity of the constriction. On the other hand the two-layer transport capacity will be increased by the action of sufficiently strong fluctuating barotropic currents at the constriction. This effect was investigated in Stigebrandt (1977).

When the supply of freshwater is relatively small and the constriction is deep and not too narrow the \( F_e \) value is small. The depth of the upper layer is then small compared to the sill depth, even if the mixing in the fjord is relatively strong. There will then be a comparatively thick intermediate layer in the fjord and \( \eta \) will attain a value near one. It seems probable that the coupling between the estuarine circulation and the circulation in the deep water is weak when the intermediate layer is thick. Fjords where \( \eta \) is near one, i.e. when the intermediate layer is thick compared to the brackish layer, may be termed N-fjords (normal fjords) and overmixed fjords may be termed O-fjords. The two regimes are indicated in Figure 2.
Note that an N-fjord may be converted to an O-fjord if the ratio of entrainment to freshwater supply is sufficiently increased. To be complete this dynamical classification should be extended to include important tidal, frictional and rotational (Coriolis) effects. How this best is done is not yet clear to the present author who first advanced these ideas in Stigebrandt (1978).

Another important result from the solution of the Stommel & Farmer equation is that there are no real solutions when $F_e$ is greater than one. When this occurs the transport is directed one way and the enclosed water body above sill level is then a freshwater lake. The estuarine Froude number, $F_e$, can be looked upon as the inverse square root of an estuarine Rayleigh number, $Rae$, defined here as

$$Rae = \frac{g \Delta \rho_o \rho_m H_m^3}{q_t^2},$$

where $q_t = Q_t / B_m$ is the volume flux per unit width of freshwater at the mouth. This quantity has the dimensions of a diffusion coefficient. The ‘Rayleigh number’ is formalized with this analogy. $\Delta \rho_o$ is the density difference between seawater and freshwater. Thus if $Rae > 1$, there may exist a two-layer flow in the constriction. A necessary condition for two-layer flow is mixing in the system.

In the following paragraphs we will construct a simple but, we hope, physically realistic model for the steady estuarine circulation. We will start with a brief recapitulation of the model for steady estuarine circulation in a wide fjord, developed by the present author (Stigebrandt, 1975). A wide fjord is particularly simple because horizontal gradients within the fjord are small. In the above-mentioned model the salinity and depth of the brackish layer are supposed to be horizontally homogeneous.

In a fjord that is not very wide there may be appreciable longitudinal gradients in the brackish layer. In order to model the longitudinal distribution of properties in the brackish layer in such a fjord the internal dynamics has to be taken into account. Long (1975) proposed a two-layer model for the circulation in an N-fjord in which the freshwater supply was concentrated at the head. In his model there are no lateral or vertical gradients in the current and salinity fields and there is no horizontal ‘turbulent’ exchange within the brackish layer. A consequence of this is that the top layer contains just pure freshwater at the head where the freshwater supply takes place. This is, however, only exceptionally observed in Norwegian fjords. As a rule the top layer is salty even at the head.

A theory for the longitudinal density gradients within the fjord is given in this paper. The theory is based on an application of Stommel & Farmer’s work to the brackish layer of N-fjords. The results compare favourably with field data from the Nordfjord.

### Properties of the brackish layer in a wide fjord

The derivations in this paragraph, mainly from a report by the present author (Stigebrandt, 1975), end up in a deterministic model for the mean estuarine circulation in wide N-fjords. In wide fjords, wide compared to the width of the mouth, the properties of the brackish layer may be nearly horizontally homogeneous and in the following we look at an ideal wide fjord, thereby completely overlooking horizontal gradients. The stratification in the fjord is thus described by a homogeneous brackish layer on top of a deep homogeneous layer of seawater. The Stommel & Farmer theory for a two-layer flow in a constriction constitutes one of the fundamentals for the theory. For an N-fjord the relation given in equation (1) degenerates to approximately $F_{a1} = 1$ or
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\[ \frac{u_{1m}^2}{g'H_{1m}} = 1, \]

where \( u_{1m} \) and \( H_{1m} \) are respectively velocity and depth of the upper layer at the mouth. Thus the tides are not supposed to affect the hydraulic control at the mouth. The upstream control implied by equation (1) is crucial for an understanding of the basic (steady) estuarine circulation in fjords.

The density field of an N-fjord is almost always determined by the salinity. An often used approximate equation of state is

\[ \rho = \rho_f (1 + \beta S), \]

where \( \rho_f \) is the density of freshwater and the salinity is expressed in \( \% \). The numerical value of \( \beta \) is about \( 8 \times 10^{-4} (\%)^{-1} \).

The mixing parameter \( P \), introduced already, is defined by

\[ P = \frac{S_2}{S_2 - S_1}, \]

where \( S_1 \) and \( S_2 \) are salinities in the upper and lower layers respectively. It is then possible to write

\[ \Delta \rho \approx \frac{\rho_2 - \rho_1}{\rho_f} = \frac{\beta S_2}{P}. \]

From the conservation of volume and salt it follows that

\[ q_{1m} = u_{1m} H_{1m} = q_r \frac{S_2}{S_2 - S_1} = P \cdot q_r, \]

where \( q_{1m} \) is the volume transport of brackish water per unit width at the mouth \( q_r = Q_r/B_m \) and \( B_m \) is the width of the mouth. From equation (2) and (5) it follows that

\[ u_{1m} = \left( \frac{Q_r \beta S_2}{B_m} \right)^{1/3}, \]

\[ H_{1m} = P \left( \frac{Q_r^2}{B_m^2 \beta S_2} \right)^{1/3}. \]

Thus we have determined the velocity and the thickness of the brackish layer in the control section at the fjord mouth as functions of the freshwater supply, the width of the constriction and the salinities of the seawater and the brackish water. Note that the velocity of the outflowing brackish water at the critical section is independent of its salinity (density) while the thickness of the brackish layer is proportional to the mixing parameter \( P \). The conditions (6) and (7) should be valid for all N-fjords with this type of control. The dynamics of the brackish water in the fjord can not change the conditions (6) and (7), but it should be remembered that \( P \) depends on the conditions within the basin. On the other hand every dynamic model for the stationary circulation of the brackish water inside such a constriction has to satisfy conditions (6) and (7) as boundary conditions at the constriction.

The parameter \( P = q_{1m}/q_r = S_2/(S_2 - S_1) \) is equal to one if there is no mixing in the fjord and \( S_1 \) is then equal to zero. We want to relate the parameter \( P \) to the mixing process that creates the mixing of seawater into the top layer. In N-fjords we expect that the mixing is mainly caused by the action of the wind on the surface. The wind-induced turbulence creates
a more or less vertically homogeneous layer at the top of the fjord. By some mechanism, the
details of which are still unknown, seawater is entrained into the brackish top layer. The
work consumed in lifting water from the lower to the middle of the upper layer per unit
time and unit surface area is

$$\omega_e \cdot g \cdot \Delta \rho \cdot \frac{1}{2} H_1,$$

where $H_1$ is the thickness of the brackish layer in the fjord and $\omega_e$ is the entrainment
velocity. The energy supplied to the fjord by the wind per unit surface area and time is

$$\tau \cdot \vec{v},$$

where $\tau$ is the vectorial wind stress and $\vec{v}$ is the surface drift velocity. Of this energy some
part will be used for effective lifting of water from the lower to the upper layer. This part is
often termed the (bulk) flux Richardson number $R_f$. Thus

$$\omega_e = \frac{2 \cdot R_f \cdot \tau \cdot \vec{v}}{g \Delta \rho H_1}.$$

By definition $|\tau| \equiv \rho u_u^2$ where $u_u$ is the so called friction velocity. The surface drift velocity
$|\vec{v}|$ should be proportional to $u_u$, thus

$$u_u = k |\vec{v}|.$$

According to Tennekes & Lumley (1972), $k \approx \frac{1}{30}$ for wall boundary layers. If $\tau$ and $\vec{v}$
are parallel we then get

$$\omega_e = \psi \cdot u_u \cdot R_i^{-1}$$

where

$$\begin{cases} 
\psi = 2 \cdot k^{-1} \cdot R_f \\
R_i^{-1} = \frac{u_u^2}{g \Delta \rho / \rho_0 \cdot H_1}.
\end{cases}$$

$R_i$ is an ordinary bulk Richardson number. The efficiency of the turbulence with respect to
working against the buoyancy forces, here termed the flux Richardson number, $R_f$, may
very well be a function of the flow parameters $u_u$, $H_1$, $g \Delta \rho / \rho_0$ and $U$, the local mean velocity
of the upper layer relative to the lower layer. Thus we expect that $R_f = R_f(u_u^2)(g \Delta \rho / \rho_0 H_1)$,
$U^2/(g \Delta \rho / \rho_0 H_1)$. Kantha & Phillips (1977) found that $\omega_e / u_u$ varied over the range of Richardson
numbers they investigated in their laboratory experiments. Several authors have determined
$R_f$ in geophysical systems. The present author for instance (Stigebrandt, 1976) obtained the
value $0.05$ in a system with breaking internal tides and other authors have reported similar
figures. With $R_f = 0.05$ we thus obtain

$$\psi = 3.$$

The often used value 2.5, derived from laboratory experiments conducted by Kato and
Phillips (see Turner, 1973), corresponds to $R_f \approx 0.04$ if $k = 1/30$. In order to make progress
in modelling the properties of the brackish layer in a wide fjord we simply assume that $R_f$ is
constant in the actual range of Richardson numbers.

We want to express $u_u$ in the appropriate wind velocity, $W$. We can express the shear as

$$\tau/\rho_w = u_u^2 = C_d \frac{\rho_u}{\rho_w} \cdot W^2,$$  \hspace{1cm} (9)

where $\rho_u$ and $\rho_w$ are the densities of air and water respectively. A commonly accepted value
of the drag coefficient, $C_d$, is $1.3 \times 10^{-3}$. This gives $(\rho_u/\rho_w C_d)^{1/2} \approx 1.25 \times 10^{-3}$. 

If the entrainment velocity does not vary over the fjord we can easily relate the mixing, \( P \), and the entrainment velocity, \( \nu_e \), because

\[
A \cdot \nu_e = Q_1 - Q_t = B_m q_m (P-1),
\]

where \( A \) is the surface area of the fjord and thus

\[
\nu_e = \frac{B_m (P-1) \cdot q_t}{A}. \tag{10}
\]

By combination of equations (8) and (10) we get

\[
P = 1 + \frac{\psi \cdot \nu_e^3 A}{B_m q_m g \Delta p / \rho_w \cdot H_1}. \tag{11}
\]

Finally by combination of equations (7), (9) and (11) and using the result \( H_1 = \phi H_{1m} \), which is discussed later, we get

\[
\frac{1}{P} = 1 - \frac{\gamma W^3 A}{\gamma W^3 A + \phi (Q_t / g \beta S_2 B_m)^2}^{1/3}, \tag{12}
\]

where \( \gamma = \psi (C_d \rho_w / \rho_w)^{3/2} \). Thus the salinity and thickness of the brackish layer in the fjord can be determined for the case of a wide fjord if the factor \( \phi \) is known.

The depth \( H_1 \) of the upper layer equation (12) may be slightly rewritten. The second term on the right is just the thickness of the brackish layer without mixing \( (P=1) \) which we write \( H_1(P=1) \). The first term on the right may be written \( \psi K \cdot L \), where

\[
L = \frac{\nu_e^3}{K q_0} \quad \text{(the Monin-Obukhov length)}
\]

and \( q_0 = g \beta S_2 Q_t / A \) is the buoyancy flux/unit surface area, \( Q_t / A \) is the specific runoff and \( K \) is von Karmans constant \((\approx 0.4)\). Thus equation (12) may be written

\[
H_1 = H_1(P=1) + \psi K L. \tag{12'}
\]

It is interesting to note that the depth of the upper layer is composed of the sum of the depth in the absence of mixing and a depth that is proportional to the Monin-Obukhov length. With \( \psi = 2.5 \) one obtains \( \psi \cdot K = 1 \). In the limit \( \nu_e \to 0 \), \( H_1 \to H_1(P=1) \) and when the mixing is strong \( H_1 \to \psi K L \). This latter result is interesting because the dynamical control in this case apparently loses its significance.

The factor \( \phi \) has been suggested to be \( 3/2 \) by, e.g., Carstens (1970). This ratio between the layer thicknesses in the fjord and at the mouth follows from the Bernoulli equation if the fjord is wide (negligible velocities in the fjord compared to those in the mouth). If dissipation occurs near the mouth the Bernoulli equation can not be used. It is then more appropriate to use the condition of conservation of momentum and from this it follows that \( \phi = \sqrt{3} \) for a wide fjord. There is no set of data that is good enough to determine whether the value of \( \phi \) is \( \sqrt{3} \) or \( 3/2 \) in natural wide fjords. It is known, however, that the salinity of the brackish water may increase during the acceleration to the critical section. This increase in salinity must be caused by turbulent activity and the turbulence most probably gets its energy from the mean current. Thus we expect that instabilities and turbulence are created near the mouth, despite the acceleration of the flow, and we believe that this is an evidence for \( \phi \) being \( \sqrt{3} \) rather than \( 3/2 \). The question of the value of \( \phi \) deserves future investigations.
The expressions (12) and (13) were derived in Stigebrandt (1975) where \( \varphi = \frac{3}{2} \) was used. Thus given the geometry of a wide fjord, the runoff and the relevant wind velocity, the salinity and the thickness of the brackish layer may be calculated from these expressions. A graphical example of the relation between the layer depth, \( H_t \), salinity, \( S_t \), freshwater runoff \( Q_t \) and the mean wind speed \( W \) is given in Figure 3 (from Stigebrandt et al., 1976) for the Nordfjord. This fjord is briefly described in the Appendix. This nomogram is constructed from equations (12) and (13).

In Stigebrandt (1975) it was shown that \( H_t \) is minimum for given mixing (wind speed) when

\[
Q_t^{5/3} = \gamma u_x^3 A \left( \frac{B_m}{g \beta S_2} \right)^{2/3}.
\]

The salinity in the brackish layer, \( S_t \), is then

\[
S_t(H_{1\text{min}}) = \frac{2}{5} S_2,
\]

where \( S_2 \) is the salinity of the lower layer. This result is independent of the value of \( \varphi \).

An interesting property of wide fjords, with zero horizontal gradients, is that the volume of freshwater stored in the brackish layer in the fjord is independent of the mixing \( P \). The volume of freshwater in the fjord, \( V_t \), is

\[
V_t = A \frac{S_2 - S_1}{S_2} H_t = \varphi A \left( \frac{Q_t^2}{g \beta S_2 B_m^2} \right)^{1/3}.
\]
The mean residence time for freshwater, $\tau_f$, is of course given by 

$$\tau_f = \frac{V_f}{Q_f}.$$

Thus the mean residence time decreases slowly with increasing runoff and is independent of the mixing.

**Conditions in N-fjords that are not very wide**

The wide fjords that were discussed in the preceding section were assumed to be horizontally homogeneous. This assumption made it possible to calculate the entrainment of seawater from below. If there are horizontal gradients in thickness and salinity of the upper layer the entrainment will have local variations even if the wind speed is constant over the fjord. There have been many attempts to construct models for the salinity and velocity distributions within N-fjords in the past. Two main branches in modelling can clearly be recognized. One branch is based on so called similarity theory. Vertical distributions of salinity and velocity are described by continuous functions. As a typical work on the similarity branch of estuarine circulation modelling we may mention the Rattray (1967) model. He considers the frictional stresses and turbulent mixing to depend on the mean circulation. Many parameters can be chosen freely and therefore the theory is physically not very convincing. Rattray does not discuss the boundary conditions at the mouth of the fjord. The concept of dynamical control is not mentioned at all.

Two layer models (also a kind of similarity models) constitute the other branch of N-fjord models. The model by Long (1975) is certainly the most cited among recent contributions. The model is conceptually fairly simple. At the head of the fjord there is freshwater supply from a river. A vertically homogeneous brackish layer occupies the upper part of the fjord and below this is the homogeneous seawater. The pycnocline is everywhere sharp. Because

![Figure 4. The vertical salinity distribution near the head in the Nordfjord (Station 1, see the map in Figure 6).](image)
of entrainment the surface water becomes less buoyant (more saline) on its way towards the
touch. [There is an interfacial stress between the brackish water and the seawater. Long
used a quadratic stress law but the drag coefficient, $K$, used by him was enormously high.
This was pointed out already in Stigebrandt (1975).] Long chooses the critical condition at the
mouth [Equation (1)] as an outer boundary condition. In Long's model there are, however,
neither lateral nor vertical velocity gradients and there is no horizontal turbulent exchange.
This implies that, e.g., salt, entrained from the lower layer, always is transported towards
the mouth of the fjord. Thus there can not be any salt in the upper layer at the head of the
fjord unless the river itself gives rise to mixing when entering the fjord. For many cases
such initial mixing is known to be weak, see McClimans (1979). Long does not discuss the
possibility of river-induced initial mixing. By experience we know, from several fjords, that
a fresh upper layer at the head of the fjord must be a rather rare event. Figure 4 shows a
typical observation from the Nordfjord where recently discharged freshwater from a local
river near the head flows relatively undiluted on top of the brackish layer. The brackish layer
below can not possibly have been generated directly by the inflowing river. This can often
easily be demonstrated from the $T$-$S$ relationships for the watermasses involved. Thus there
must be some mechanism in the fjord that performs an inward transport of salt in the brackish
layer. There are several mechanisms for such a transport.

(i) Overturning in the brackish layer caused by a longitudinal density gradient. When
this occurs, the upper brackish layer can not be vertically homogeneous. Hydraulically
this requires a subdivision of the brackish layer.

(ii) Horizontal transport caused by the mean flow itself, the wind and possibly also by
the tides. Such mechanisms are widely claimed in oceanographic modelling. When
the details of the physical processes are not available or understood the processes are
commonly modelled as diffusion.

(iii) The fjord may be so wide that the outgoing brackish current does not fill the whole
width of the fjord. The Coriolis force keeps the outgoing current at the right fjord
bank. The control mechanism is in this case not as simple as the one expressed in
equation (2) (see Whitehead et al., 1974). We do not consider these wide, rotating
fjords in this paper.

In the following we will investigate the effect the mechanism suggested in (i) has in
controlling the salinity distribution along the longitudinal fjord axis.

We may look upon the problem from a slightly different point of view. In Long's model
vertical columns of brackish water are forced to stay vertical on their way through the fjord.
The water in the columns is initially fresh at the head. During the advection through the
fjord the water in the columns becomes more and more saline because of entrainment of
seawater from below. This pictured situation with a fresh upper layer at the head is seldom
observed, at least in Norwegian fjords, and thus this description of the flow is not generally
correct. Thus the water is in many cases not moving through the fjord as if it were contained
in vertical columns. The columns are now and then overturning and the overturning can be
expected to occur when the longitudinal density gradients become too strong. We know that
in a medium at rest no horizontal gradients of density are possible. If there is horizontal
advection in the system, however, there may also be horizontal gradients. We want to find a
mechanism that regulates the magnitude of the horizontal gradients.

Stommel & Farmer (1953) taught us that there is an upper limit for the salinity of the top
layer in a constricted basin with given freshwater runoff. There is thus a smallest possible
difference between the salinity of the seawater outside the constriction and the salinity of the
top layer inside. As pointed out in the introduction this smallest possible salinity difference is
reached when the two-layer transport capacity of the constriction is fully utilized. The largest possible density difference, however, is of course reached when the constricted basin contains just pure freshwater. This state is reached when the estuarine Froude number, $F_e$, is equal to one, see the introduction.

We will now extend this latter result to the brackish layer of the N-fjord. A natural fjord does not usually have a constant width. The fjord can be looked upon as made up by a series of basins connected to each other by more narrow transitions. We are, for simplicity, assuming that the brackish layer of each basin is homogeneous with respect to density. The question then is: What is the largest possible density difference between the two neighbouring sub-basins?

The depth of the pycnocline is primarily determined by the critical condition at the fjord mouth, equation (2). Two neighbouring sub-basins will have nearly equal thicknesses of their brackish layers if the differences in their salinities is much less than the vertical difference. If we consider the pycnocline as dynamically passive, like a rigid porous bottom, we can directly use the result $F_e = 1$ to determine the largest possible steady-state density difference between the brackish waters in the sub-basins given the net volume flux ($P_s Q_f$) through the constriction. $P_s$ is the value of the mixing parameter $P$ in the inner of the two basins. The relevant density difference is now the density difference between the brackish waters in the two basins. The modified estuarine Froude number we call the internal estuarine Froude number, $F_{el}$. If $F_{el}$ by some reason, e.g. heavy mixing in the outer of the two basins because of larger basin area and/or stronger winds, drops below one there will be a two-way flow in the brackish layer in the constriction. Saltier water from the outer basin will flow into the inner basin until the density difference has been so small that $F_{el}$ attains the value one again. This mechanism should be able to control the salinity (density) difference between two neighbouring basins and thus determine the horizontal salinity distribution in the fjord. Thus our hypothesis is that this mechanism tends to keep $F_{el}$ near one.

Under these conditions the salinity jump, $\Delta S_s$, over the secondary constriction (separating the two sub-basins) is given by

$$F_{el}^{-2} = Ra_{el} = \frac{g \beta \Delta S_s P_s H_{1s}^3}{P_s^2 Q_f^2} \approx 1,$$

where $\beta \Delta S_s = \Delta \rho_2 / \rho_0$ and $H_{1s} =$ depth to the pycnocline in the constriction. As noticed in the introduction the square of the estuarine Froude number can be looked upon as an inverted (estuarine) Rayleigh number. Flow against the source (the river at the head) occurs in the brackish layer when the Rayleigh number is greater than critical (one in this case). The critical value of the appropriate Rayleigh number controls the convection and thereby the horizontal salinity (density) distribution.

The depth to the pycnocline, $H_1$, in the outer part of the fjord is determined by the hydraulic control at the mouth and from equation (7) we get, using the factor $\varphi$ introduced in section 2.

$$H_1^3 = \varphi^3 P_0 \frac{Q_t^2}{B_m \frac{g \beta S_s}{3}}.$$

The thickness of the upper layer in every sub-basin is supposed to be proportional to the thickness in the basin nearest the mouth, thus

$$H_{1s} \propto H_1.$$
From equation (14) we then obtain

\[ \Delta S_1 \propto (S_2 - S_1) \frac{1}{\varphi^3} \frac{(P_s)^2}{(B_m)^2}. \]  

(14)

The salinity difference between, e.g., the mouth and the head, \( \Delta S_1 \), should be

\[ \Delta S_1 = \sum_{\text{all constrictions}} \Delta S_s. \]

We assume that \( P_s \) and \( P \) are proportional and then we get

\[ \Delta S_1 = \alpha (S_2 - S_1) = \frac{a S_2}{P} \]  

(15)

where \( \alpha \) is a constant with a unique value for each fjord. The value of \( \alpha \) can hardly be determined theoretically, it has to be determined from field measurements.

From considerations of the two-layer transport properties of the brackish layer in constrictions we have thus reached the conclusion that the density (salinity) difference between mouth and head should be inversely proportional to the mixing, \( P \), as measured at the fjord mouth. This conclusion is the opposite of the result of the theory by Long (1975) where the density difference \( (\Delta S_1) \) increases with \( P \). Of course \( \Delta S_1 = S_1 \), as we do not tolerate negative salinities, and this means that, for fjords with heavy runoff and/or weak mixing, equation (15) has to be replaced by just \( \Delta S_1 = S_1 \). Long’s theory has this property and it may therefore be valid for narrow fjords with heavy runoff and/or weak mixing.

The predictions of the theory will be compared to field measurements from the Nordfjord. The Nordfjord and the measurements from it are described in the Appendix but we will already here present some results derived from a series of measurements, obtained by the Geophysical Institute at the University of Bergen, Norway, during the years 1932–66. During this period standard hydrographic measurements were taken along the fjord axis, see Figure 5. We will use one station at the mouth, Bryggja, and one near the head, Utvik. These old measurements were analyzed by Stigebrandt et al. (1976) and from their report the following information is taken. The runoff to the Nordfjord has a pronounced yearly variation with a maximum in the summer and a minimum in the winter. The measurements from each month of the year were grouped together and mean values were calculated. These

![Figure 5. Map of the Nordfjord.](image-url)
mean values are presented in Table 1 below. Here $S_2=$ salinity ($\%_o$) at 20 m depth, $S_1=$ salinity ($\%_o$) at the surface at Bryggja and $\Delta S_1=$ salinity difference ($\%_o$) at the surface between Bryggja and Utvik.

During the winter months there may be appreciable density effects of temperature and for other purposes this fact should be considered. In Figure 6 $\Delta S_1$ is plotted vs. $S_2-S_1$. A straight line is fitted by eye. This line is equation (15) with $a=0.65$. The correlation between the straight line and the data points is quite good. From this we may infer that the theory predicts quite well the observations from the Nordfjord. The hypothesis that $F_{el}$ in the transitions between the sub-basins tends to approach a critical value seems to be true. It is hard to see how any of the other mechanisms, suggested under (ii) before, can produce this density difference variation. From Figure 6 can be seen that the 'monthly mean Nordfjord' does not reach the range with high runoff and/or weak mixing where $\Delta S_1=S_1$. In Figure 6 there is also drawn a continuous curve through the data points. It can be seen that for periods with increasing discharge $\Delta S_1$ is larger and for periods with decreasing discharge it is lower than predicted by the theory. The $\Delta S_1$ value for September seems to be a little bit high but actually there is a weak secondary maximum in runoff at the end of this month. The hysteresis displayed by the continuous curve is certainly caused by the variation in the freshwater supply together with the inertia of the system, the brackish layer having a finite time constant (on the order of weeks).

**Table 1. Salinity difference between the seawater and the brackish water at the mouth ($S_1$--$S_2$) and salinity difference in the surface water between mouth and head ($\Delta S_1$) in the Nordfjord. Monthly mean values**

<table>
<thead>
<tr>
<th>Month</th>
<th>January</th>
<th>February</th>
<th>March</th>
<th>April</th>
<th>May</th>
<th>June</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1-S_1$</td>
<td>1.34</td>
<td>0.73</td>
<td>0.90</td>
<td>4.36</td>
<td>9.72</td>
<td>14.52</td>
</tr>
<tr>
<td>$\Delta S_1$</td>
<td>-0.12</td>
<td>-0.13</td>
<td>2.68</td>
<td>3.59</td>
<td>6.95</td>
<td>11.90</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Month</th>
<th>July</th>
<th>August</th>
<th>September</th>
<th>October</th>
<th>November</th>
<th>December</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1-S_1$</td>
<td>18.22</td>
<td>18.92</td>
<td>13.11</td>
<td>8.43</td>
<td>5.48</td>
<td>3.89</td>
</tr>
<tr>
<td>$\Delta S_1$</td>
<td>12.30</td>
<td>10.23</td>
<td>8.72</td>
<td>4.10</td>
<td>2.93</td>
<td>0.20</td>
</tr>
</tbody>
</table>

**Figure 6. Salinity difference in the brackish layer between the mouth and the head, $\Delta S_1$, vs. the salinity difference between the brackish layer at the mouth and the underlying seawater, $S_1-S_2$. The crosses represent measured monthly mean values in the Nordfjord. The straight, continuous line, fitted by eye, represents the expected relationship when $\Delta S_1=0.65$ ($S_1-S_2$). The continuous curve, connecting the crosses, is the fjord salinity loop.**
We may consider the findings made here from a more philosophical point of view. The existence of an internal dynamical control in a system with buoyancy sources and sinks and with mixing implies that no boundary conditions upon the stratification can be put on the system upstream of the control. The stratification can thus not be prescribed anywhere, it has to be calculated everywhere in the system.

To justify the assumption of longitudinal homogeneity made in section 2 we notice that in a wide fjord $B_s$ is everywhere much greater than $B_m$ and therefore, from equation (14'), $\Delta S_s$ is particularly small in this kind of fjord. This explains why sufficiently wide fjords may be treated as longitudinally homogeneous with respect to properties in the brackish water.

**Discussion**

The simple mechanism for salinity difference control between two neighbouring sub-basins in a fjord with freshwater runoff and dynamical control at the mouth presented in the last section seems to explain the observed density (salinity) difference between head and mouth in the Nordfjord. The discrepancy between theory and observations could be qualitatively explained in terms of changing freshwater runoff to the fjord and inertia of the brackish layer. The theory should most certainly apply to many fjords with dynamical control at their mouths. The assumption of a fjord basin composed of several sub-basins connected by narrower transitions, although nearly always satisfied in nature, does not seem to be a crucial assumption. A similar critical internal Rayleigh number should exist even with a constant width of the fjord. This Rayleigh number should be based upon the horizontal density gradient instead of the density jump between two neighbouring sub-basins and possibly also be influenced by frictional effects.

The theory for a wide fjord presented in section 2 may be used to make a quick first-order determination of the thickness and salinity of the brackish layer just within the mouth of the fjord. The theory does not contain any adjustable parameters. After the determination of the salinity of the brackish layer at the mouth has been performed the salinity at the head (or elsewhere in the fjord) can be determined from the theory of section 3 if only the fjord constant, $a$, has been determined. These two theories together should give the mean properties of the brackish layer in the fjord.

The final step in modelling the estuarine circulation in fjords is to construct a model which gives the actual velocity and density profiles in every point of the fjord given the wind field, the freshwater runoff, the tides and the vertical stratification in the sea outside the fjord. This aim seems today almost unattainable. For example short period fluctuations in wind speed (short compared to the residence time for the freshwater in the fjord) introduce tremendous difficulties in the modelling of short time behaviour of the fjord, mostly because of the changing mixing conditions. Meanwhile we can do a lot to improve the models for the mean estuarine circulations in N-fjords outlined in this paper. This seems to be a sound way to go because these models apparently contain the basic physical processes that govern the estuarine circulation in N-fjords.

**References**

Appendix

The Nordfjord has a length of slightly more than 100 km and the surface area is about 300 km². A great part of the fjord has depths between 200 and 500 m. The sill depth is about 100 m and the sill is situated outside the fjord mouth. The fjord has its main connection with the sea between the islands Vågsvågøya and Husevågøya but there are some narrow and shallow connections with the sea between the Vågsvågøya and the mainland in the north and the Bremangerland and the mainland in the south and between the Bremangerland and the Husevågøya, see the map in Figure 5. The effective width at the mouth, \( B_m \), is approximately 1200 m.

The annual mean of freshwater runoff to the fjord is about 200 m³ s⁻¹, the dominating part is released in the inner reaches of the fjord. There is a pronounced yearly variation in the runoff with a maximum monthly mean of about 400 m³ s⁻¹ in June and July and a minimum monthly mean of about 30 m³ s⁻¹ in February. The mean wind speed, \( \bar{W} \), (cubic root of mean cube wind speed) is typically 5-6 m s⁻¹.

Many standard hydrographic measurements have been undertaken in the fjord. During the years 1932-66 about 140 hydrographical sections were obtained along the fjord axis. Saelen (1967) has used some of these data to investigate the exchange of deep water in the fjord. Stigebrandt et al. (1976) used all the old data and even some new in order to make predictions of the effect of increased freshwater discharge in the winter season on the ice conditions in the fjord. It was found in the cited report that for freshwater discharges less than about 200 m³ s⁻¹ the Nordfjord follows quite well the theory for a wide fjord, outlined in section 2 in this paper. For greater discharges the horizontal gradients become pronounced but the depth of the brackish layer is still well predicted by the wide fjord theory. A detailed analysis of the estuarine circulation in the Nordfjord will appear elsewhere.