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Abstract—Results from numerical calculations of wind-driven ocean circulation with a barotropic model with bottom friction and on the \( \beta \)-plane are presented for a range of parameters from slightly non-linear to highly non-linear flows. The imposed wind-stress curl is steady and everywhere negative and is symmetric with a maximum amplitude at mid-latitudes. The oceanic response is strongly asymmetric with a westward intensification for nearly linear flows (STOMMEL's 1948 result), and a northward concentration for non-linear flows. All of the cases showed a steady oceanic response to a steady wind. The effect of the non-linear processes is to decrease the transport below the Sverdrup transport for mildly non-linear flows. Strongly non-linear flows in which the western boundary current flows up to the northern boundary, across the ocean and southward along the eastern boundary have transports larger than the Sverdrup transport. It is shown that non-linear effects cause the western boundary current to flow northward into regions where there is no wind-stress. Thus when flows are non-linear it appears that one must consider entire oceanic basins rather than basins delimited by the sign of the wind-stress curl. The balance of forces in the different regions of the ocean are presented and the feasibility of constructing analytic models is discussed. Details such as the countercurrent on the off-shore side of the western boundary current are reflected in the associated pressure fields which are also graphed.

1. INTRODUCTION

This second part of the paper on wind-driven ocean circulation is concerned with numerical solutions obtained for the non-linear finite-difference equations of the barotropic model which was introduced in Part 1 (VERONIS, 1965a). We have already looked at non-linear effects by means of a perturbation method whose validity requires that the thickness of the boundary layer which is controlled by inertial effects be less than that of the frictional boundary layer of the linear problem. This condition is relaxed in the present part of the paper and we shall consider flows which have non-linear effects of varying degrees. Our primary interest is, of course, in those flows which are strongly non-linear and are not tractable by means of a perturbation method or by boundary-layer techniques.

It was pointed out in the introduction to Part 1 that many of the results of Part 1 were implicitly or explicitly contained in the papers by STOMMEL (1948) and MUNK, GROVES and CARRIER (1950). The results of the present part, on the other hand, have not been derived previously, although BRYAN (1963) has presented numerical solutions for moderately non-linear flows. Bryan's calculations were made with a finer finite-difference network than those of the present paper. This enabled him to discuss cases with more realistic values of the parameters than those used here. However, runs were made in the present work over a sufficient range.
of parameters so that the asymptotic behaviour of the system was strongly indicated by the similarity of results for different values of the parameters. Also, the method of attack made it possible to consider very strongly non-linear flows.

Analytical attempts on the non-linear problem have been (understandably) more restrictive in scope than the present work. Fofonoff (1954) made a significant contribution in his study of free inertial flows. Our results in section 3 show that flow patterns similar to his are, in fact, generated when inertial forces are large even though wind-stress and friction are necessary features in our derived flows.

Stommel produced an early model of an inertially controlled Gulf Stream and later published the results in his book (Stommel, 1958). Charney (1955) and Morgan (1956) gave more detailed treatments of both barotropic and baroclinic inertial models of the Gulf Stream. These three papers contained the assumption that only the incipient formation region (say, from the Florida Straits to Cape Hatteras) could be studied by an inertially controlled model. Morgan made the assumption explicit and supposed that friction and transient processes could be "postponed" to the northern regions. The results of this paper support the basic assumption of these models.

A more ambitious attempt was undertaken by Carrier and Robinson (1962) when they tried to discuss the entire circulation on the basis of boundary flows which are dominated by inertial processes. Moore (1963) demonstrated that the basic assumptions of Carrier and Robinson was incorrect for flows with a Navier-Stokes law for frictional processes. The present work adds further evidence (if such is needed) to Moore's demonstration. Here we treat frictional processes through a bottom drag (sometimes termed Ekman-friction) law.

The flaw in the reasoning of Carrier and Robinson is that they assumed that flow in an inertial boundary layer could be connected to a Sverdrup interior region without passing through a region where friction is dominant. The result of this assumption is that a western boundary layer cannot exist in regions where the northward gradient of wind-stress curl is negative. In all of our moderately non-linear flows a boundary layer exists along the entire western boundary. In strongly non-linear flows the boundary-layer structure is most intense in the north and the transition from an inertially dominated boundary layer to a Sverdrup interior does not take place.

In the next section the non-linear partial differential equations for the barotropic model are approximated by a set of finite-difference equations. The method of integration is outlined and some information pertinent to the numerical solution is presented. The remainder of the paper is concerned with a discussion of the different types of solutions.

2. EQUATIONS AND METHODS OF INTEGRATION

The equations for the study of wind-driven ocean circulation as described by vertically averaged quantities for an ocean of constant mean depth on the β-plane are (in non-dimensional form)*

\[
\frac{\partial u}{\partial t} + R v \cdot \nabla u - f v u = \frac{\partial p}{\partial x} - \epsilon u + \tau^d
\]  

(2.1)

*See Part 1 for a detailed discussion of the non-dimensionalization of the system.

\[ \frac{\partial v}{\partial t} + R \mathbf{v} \cdot \nabla v + f \mathbf{u} = - \frac{\partial p}{\partial y} - \epsilon v + \tau^y \quad (2.2) \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.3) \]

where \( \mathbf{v} \) is the velocity vector with components \( u \) taken position eastward (x) and \( v \) taken positive northward (y); \( f = \frac{a}{L} \tan \phi_0 + y \) is the Coriolis parameter where \( a \) is the radius of the earth, \( \pi L \) is the east-west and north-south dimension of the basin and \( \phi_0 \) is the tangent latitude for the \( \beta \)-plane model; \( x \) and \( y \) have been non-

\[ \text{dimensionalized with scale } L \text{ so that } x \text{ and } y \text{ range from } 0 \text{ to } \pi; \]

\[ p \] is a non-dimensional pressure; the time, \( t \), is non-dimensionalized with respect to \( \beta L \); \( \beta = \frac{2\Omega}{a \cos \phi_0} \), where \( \Omega \) is the rate of rotation of the earth; and \( \tau^x \) and \( \tau^y \) are the normalized components of the wind-stress.

The frictional parameter, \( \epsilon \), and the Rossby number, \( R \), measure the effects of (bottom) friction and non-linearity respectively and are defined by

\[ \epsilon = \frac{K}{\beta L}, \quad R = \frac{W}{\rho \beta^2 L^3} \quad (2.4) \]

where \( K \) is the coefficient of bottom friction, \( W \) is the amplitude of the wind-stress, and \( D \) is the depth of the barotropic ocean.

The continuity equation (2.3) admits the introduction of a stream function defined by \( u = -\psi_y, \ v = \psi_x \). Elimination of pressure by cross differentiating (2.1) and (2.2) yields the vorticity equation

\[ \frac{\partial \zeta}{\partial t} + R \mathbf{v} \cdot \nabla \zeta + v = -\epsilon \zeta + \text{curl } \tau \quad (2.5) \]

where

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi \quad (2.6) \]

is the vertical vorticity component. We wish to solve equation (2.5) together with the boundary condition that \( \psi = 0 \) on \( x = 0, \pi \) and \( y = 0, \pi \).

The above system of equations has been non-dimensionalized so that in the vorticity equation the coefficient of the term arising from the variation of the coriolis parameter and the coefficient of the wind-stress curl are both one. When these two terms balance and are dominant, the solution is the SVERDRUP (1947) solution. It forms a good pivot point for the discussion of all of the numerical solutions. It is described in detail in Part 1.

The numerical integrations are carried out for the vorticity equation (2.5) which, together with (2.6) and the boundary conditions, forms a closed system for \( \psi \). Hence, at any time \( t \) the quantities \( \zeta, \psi, u, \) and \( v \) can be determined. We can take the divergence of equations (2.1) and (2.2) to derive a Poisson equation for \( p \) which can then be relaxed for the pressure field.

To carry out the integration of equation (2.5) we rewrite the non-linear terms, \( \mathbf{v} \cdot \nabla \zeta \), in the more convenient equivalent form

\[ \mathbf{v} \cdot \nabla \zeta = \nabla \cdot (\mathbf{v} \zeta) \quad (2.7) \]
Let the ocean basin be divided into a network of \( N - 1 \) grid intervals with spatial intervals \( \Delta x = \Delta y = \frac{\pi}{N-1} \) as shown in Fig. 1. The quantity, \( q_{ij} \), corresponds to the \( v \) component of \( q \) at the point \( x = (i - 1) \Delta x, y = (j - 1) \Delta y \). Derivatives are written as finite differences by means of the centered difference scheme:

\[
\frac{\partial q}{\partial x} = \frac{q_{i+1,j} - q_{i-1,j}}{2 \Delta x}, \quad \frac{\partial q}{\partial y} = \frac{q_{i,j+1} - q_{i,j-1}}{2 \Delta y}
\]

Time is divided into increments, \( \Delta T \), and we write \( t = n \Delta T \) where \( n = 0, 1, 2 \), so that \( q(t) \) is denoted by \( q^n \). Then

\[
\frac{\partial q}{\partial t} = \frac{q^{n+1}_{ij} - q^{n-1}_{ij}}{2 \Delta T}
\]

With these definitions we rewrite (2.5) in the form

\[
\frac{\xi^{n+1}_{ij}}{\Delta t} = \frac{R \Delta T}{2 \Delta x \Delta y} J (\psi^n, \xi^n) - \frac{\Delta T}{\Delta x} (\psi^{n+1}_{i+1,j} - \psi^n_{i-1,j})
\]

where

\[
J (\psi^n, \xi^n) = (\psi^{n+1}_{i+1,j+1} - \psi^n_{i-1,j+1}) \xi^n_{i+1,j} - (\psi^{n+1}_{i+1,j-1} - \psi^n_{i-1,j-1}) \xi^n_{i+1,j-1}
\]

\[= (\psi^{n+1}_{i+1,j} - \psi^n_{i+1,j}) \xi^n_{i,j+1} + (\psi^{n+1}_{i-1,j+1} - \psi^n_{i-1,j+1}) \xi^n_{i-1,j+1}
\]

and

\[
\xi^n_{ij} = \frac{1}{\Delta x \Delta y} (\xi^{n+1}_{i+1,j} + \xi^{n+1}_{i,j+1} + \xi^{n+1}_{i,j} + \xi^{n+1}_{i-1,j} - 4 \psi^n_{ij})
\]

The frictional term \( \epsilon \xi \) has been approximated by \( \frac{\epsilon}{2} (\xi^{n+1}_{ij} + \xi^{n-1}_{ij}) \) and discussed below in subsection (a).

![Fig. 1. The network of \( N - 1 \) grid intervals in each of the \( x \) and \( y \) directions into which the ocean basin is divided for the numerical study. The points are spaced \( \frac{\pi}{N-1} \) units apart.](image)

(a) Stability considerations

In any numerical integration of a set of finite-difference equations a serious consideration is the stability of the numerical system. There are many discussions of this problem in the literature and we shall not go into it in any detail here. Instead, we simply state the numerically stable method which was used.
The form of the Jacobian, $J(\psi^n, \zeta^n)$, which was used has two advantages. In the first place it conserves both vorticity (corresponding to $\int_0^\pi \int_0^\pi v \cdot \nabla \zeta \, dx \, dy = 0$) and kinetic energy (corresponding to $\int_0^\pi \int_0^\pi \psi v \cdot \nabla \zeta \, dx \, dy = 0$). Secondly, at points next to any boundary the Jacobian can be evaluated directly because, whenever a boundary value of $\zeta$ appears, it is multiplied by boundary values of $\psi$ which vanish. Hence, it is never necessary to evaluate $\zeta$ on the boundary. If it were, we would be forced to take one-sided derivatives, a procedure which usually leads to computational instability.

The frictional term, $\epsilon \zeta$, has been evaluated as the average of the values of $\zeta$ at one time-step in the past and one time-step in the future. The reason for that is that such an implicit evaluation of the frictional term leads to a numerical integration procedure which is always stable (at least, with respect to the frictional term) and since it introduces no added difficulty we make use of this form.

Centered time-differencing will eventually lead to computational instability of an evolving system. In the present integrations the systems always evolved to a steady state and no instability occurred. For transient systems it will probably be necessary to use a different type of time integration.

(b) **Redefined variables**

To minimize the number of operations we absorb some of the coefficients into the dependent variables. Thus we write:

$$\psi^n_{ij} = \frac{2}{\Delta T} \frac{1 + \epsilon \Delta T}{1 + \epsilon \Delta T} \Delta x \Delta y \frac{R}{\Delta T} \xi^n_{ij}, \quad \xi^n_{ij} = \frac{2}{\Delta T} \frac{1 + \epsilon \Delta T}{1 + \epsilon \Delta T} R \zeta^n_{ij}$$

and in terms of the capped variables the equations become

$$\zeta^{n+1}_{ij} = \theta \zeta^{n-1}_{ij} - J(\hat{\psi}^n, \xi^n) - \eta (\hat{\psi}^{n+1}_{i+1,j} - \hat{\psi}^{n+1}_{i-1,j} - \hat{\psi}^{n+1}_{i,j+1} + \hat{\psi}^{n+1}_{i,j-1}) - \hat{\zeta}^{n-1}_{ij} + R (\text{curl} \tau)^n_{ij} \quad (2.14)$$

where

$$\theta = \frac{1}{1 + \epsilon \Delta T}, \quad \eta = \frac{\Delta T \Delta y}{1 + \epsilon \Delta T}, \quad \hat{\epsilon} = \frac{\epsilon \Delta T}{1 + \epsilon \Delta T}, \quad \hat{R} = \frac{R (\Delta T \Delta y)^2}{(1 + \epsilon \Delta T)^2} \quad (2.15)$$

and

$$\zeta^n_{ij} = \hat{\psi}^{n+1}_{i,j+1} + \hat{\psi}^{n+1}_{i,j-1} + \hat{\psi}^{n+1}_{i+1,j} + \hat{\psi}^{n+1}_{i-1,j} - 4 \hat{\psi}^{n+1}_{i,j} \quad (2.16)$$

Once the system is described in this fashion so that only coefficients equal to one appear in the Poisson equation (2.16) and in front of $\hat{\zeta}^{n+1}_{ij}$ and $J(\hat{\psi}^n, \xi^n)$ in equation (2.14), the original method of scaling and non-dimensionalizing is immaterial because the same coefficients always appear in (2.14). In this sense, then, no discriminatory scaling information is left in the finite-difference equations.

(c) **The Poisson equation**

Integrating equation (2.14) on an electronic computer is now straightforward. However, at each time-step it is necessary to solve (2.16) for $\psi$. Two methods were tried. The first was to relax for $\psi$ using the accelerated Liebmann method. It turned out that the truncation error associated with the relaxation gave rise to a spurious oscillation of the system for all the reasonable (with regard to length of computing
time) convergence criteria attempted. Hence, the equation was solved exactly by the following method.

Let

\[ \dot{\phi}_{n,ij} = \sum_{k=1}^{N} A_{ik} \sin \frac{k\pi (j - 1)}{N} \]

where \( N \) is the number of grid intervals. Here, \( \dot{\phi}_{n,ij} \) automatically satisfies the boundary conditions at \( j = 1 \) and \( j = N \). It is a straightforward procedure to derive the following difference equation for the \( A_{ij} \) from (2.16) and (2.17):

\[ A_{i+1,j} = \left( 4 - 2 \cos \frac{k\pi}{N} \right) A_{ij} - A_{i-1,j} + \frac{2}{N} \sum_{j=1}^{N} \sin \frac{k\pi (j - 1)}{N} \dot{\phi}_{ij} \]

This difference equation together with the boundary conditions is solved by the method discussed in Richtmyer (1957). The field can now be evaluated exactly from the \( A_{ij} \) and equation (2.17). The method is exact for the finite difference system.

The major part of the computing time for the integration of the finite-difference system was spent on the solution of the Poisson equation.

(d) Numerical procedure

We close this section with a discussion of the philosophy which determined the method of attack used to derive the numerical solutions. There are two parameters, \( \epsilon \) and \( R \), in the set of equations which determine the behaviour of the system. All of the physical features obtainable from the basic model are inherent in the system and can be studied by looking at the results of integrations of the model for ranges of values of \( \epsilon \) and \( R \).

If the model were based on a straightforward derivation of a set of equations which is known to be an approximation (in some sense) of the equations which determine the observed flow in the ocean and if the associated parameters were well-defined, then we would be faced with a straightforward task. Specifically we would have to integrate the derived set of equations for some definite values of \( \epsilon \) and \( R \). However, the barotropic model is not an approximation to the real ocean in a way which is a priori obvious and the "real" values of \( \epsilon \) and \( R \) are not known.

What we can do is to treat simplified models such as the steady, linear, frictionally controlled model and deduce that, if the ocean circulation is governed by the dynamics of the simplified model, the value of \( \epsilon \) must be bounded by a certain extreme value. It was shown in an earlier paper (Veronis, 1965) that with such an approach an upper bound for \( \epsilon \) is to be it order of 0.01. A similar argument for an inertially-controlled model indicated that \( \sqrt{R} \lesssim 0.01 \). Hence, ideally we would like to study flows with both \( \epsilon \) and \( \sqrt{R} \) individually ranging from values of 0.01 to smaller values.

Such an approach is not feasible with present computing facilities because of both the length of time required for the integrations and the capacity of the high-speed core memories. It would be necessary to treat cases with at least 10,000 grid points (and probably 40,000) and to carry out the time integrations until a steady state is approached. Instead, we have chosen a smaller finite-difference network (cases were run with 400, 900, and 1600 grid points). The values of \( \epsilon \) and \( \sqrt{R} \) were adjusted so that the scale of the processes determined by \( \epsilon \) and \( \sqrt{R} \) were sufficiently
large so as to be describable by the network used. Runs were made with differently sized grids to determine what grid-size was necessary to yield reliable results for specifically chosen values of $\epsilon$ and $\sqrt{R}$. By "reliable" we mean that the vorticity at each point agreed (to within 5%) with the vorticity derived by using a finer grid. When such a criterion was established, production runs were made for the desired ranges of parameters.

In Part 1 both frictional and inertial boundary-layer scales were established. Specifically from the linear, frictional, boundary-layer analysis we found that $\epsilon/\pi$ represented a convenient measure of the boundary-layer scale. The inertial boundary-layer was similarly found to be represented by the value $\sqrt{R}/\pi$. The numerical calculations showed that it is necessary to have only one point in the boundary layer defined in this manner in order to ensure that the results are reliable. For example, for the frictional case with $\epsilon = 0.1$, a $30 \times 30$ grid is adequate. Analysis of the case $\epsilon = 0.1$ with a coarser grid is accompanied by some distortion of the vorticity field. It was also found that it was sufficient to apply this "reliability" criterion to the coarser of the two scales. Hence, if the inertial boundary layer were thicker than the frictional one, it sufficed to base the criterion on the inertial boundary-layer thickness.

Figure 2 shows the distribution of the values of $\epsilon$ and $\sqrt{R}$ for the cases which have been run. Points with $\epsilon \leq \sqrt{R}$ correspond to flows in which the inertial
boundary layer is thicker than the frictional boundary layer. It is this region which is of greater interest to us because it is not accessible by analytical methods.

3. Discussion of Numerical Solutions

The pertinent dimensional parameters have the values

\[ \Omega = 0.73 \times 10^{-4} \text{ sec}^{-1} \]
\[ L = 2 \times 10^8 \text{ cm} \]
\[ a = 6.4 \times 10^8 \text{ cm} \]
\[ \phi_0 = 30 \]
\[ \beta = 2 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1} \]

The numerical integrations were all carried out in the same way. At time \( t = 0 \) with initial conditions \( \psi = 0 \) everywhere a wind-stress curl of the form

\[ \text{curl } \tau = - \sin x \sin y \]

was introduced. With this steady value for curl \( \tau \) the system gradually evolved from an initial state until a steady state was achieved. In every calculation the flow eventually became steady.* A typical approach to the steady state is shown in Fig. 3 where the kinetic energy of the entire flow field is plotted against time for the case \( \epsilon = 0.05, \sqrt{R} = 0.1 \). For practical purposes convergence or steadiness was usually achieved after a time given approximately by \( t \approx 7/\epsilon \).

Some of the steady-state solutions are exhibited in Figs. 4 to 9. After the system settled to a steady state, contour plots were made of the streamline pattern, the pressure field and the vorticity pattern. The contour intervals are in units of one-fifth of the maximum absolute value of the variable. For example, in the case of the stream function (which is always positive) the streamline contours correspond to \( \psi/\psi_{\max} = 0, \psi/\psi_{\max} = 0.2, \text{ etc.} \), up to \( \psi/\psi_{\max} = 1.0 \). Maximum values of the vorticity are concentrated near the western boundary for the more linear solutions (smaller \( R \)); hence, the contour lines are crowded together in that region.

(a) Linear system

Figure 4 shows \( \psi, p, \) and \( \zeta \) computed with a \( 40 \times 40 \) grid for the case \( \epsilon = 0.05, \sqrt{R} = 0.001 \). The reliability criterion which was discussed in the previous section is marginally satisfied for this case (and for the following two). This case is practically linear and the various fields could as well have been plotted from the analytical solution given in Part 1. The principal points which we wish to stress here are:

(a) the stream function is symmetric about mid-latitude; (b) the isobars nearly parallel the streamlines so that the flow is nearly geostrophic; (c) the vorticity is symmetric about mid-latitude and is highly concentrated near the western boundary.

(b) Non-linear systems

Results are shown in Fig. 5 for the case where the inertial boundary-layer thickness is six-tenths of the frictional boundary-layer thickness, i.e., \( \epsilon = 0.05 \).

*By "steady" we mean that the vorticity at every point of the grid differed by less than 1% from its value at the previous time step. This criterion is probably too stringent and about one third of the computer time could be saved by a less stringent criterion.
Fig. 4. Contours of $\psi$, $p$ and $\zeta$ (in that order reading down) for the case $\epsilon = 0.05$, $\sqrt{R} = 0.001$. In each graph the pertinent variable is normalized with respect to its maximum value. The pressure field can be determined only to within a constant and the latter has been chosen so as to give the minimum $p = 0$. $\psi$ is everywhere positive and five contours are shown spaced in units of 0.2 from $\psi/\psi_{\text{max}} = 0$ to $\psi/\psi_{\text{max}} = 1.0$. Similar remarks obtain for $p$. The vorticity is almost everywhere negative with the maximum negative value, indicated by a small circle, occurring at mid-latitude along the western boundary. The region of positive vorticity is shaded. The point of minimum $p$ also occurs along the western boundary but toward the north. $\psi_{\text{max}} = 1.841$ and $|\zeta_{\text{max}}| = 233.6$. 
\( \sqrt{R} = 0.03 \). Non-linear effects are clearly present and they correspond to the results derived in Part 1 by a perturbation treatment of the nonlinear terms in the equations. The western boundary layer in the southern part of the ocean is broader than that of the linear system, i.e., the streamlines turn northward sooner (at a larger value of \( x \)) in the more nonlinear case. In the northern half-basin the western boundary layer shows a more concentrated structure of northward flow and before the streamlines enter the interior, linear (Sverdrup-like) domain there is a boundary-layer region of alternating northward and southward flow with an accompanying positive vorticity. These features are reflected in the isobar pattern which again nearly parallels the streamline pattern. The principal modification of the linear flow in the pressure field is that, where inertial effects are significant, the non-geostrophic flow tends to be down the pressure gradient. In the linear system the wind-stress forced flow up the pressure gradient. The vorticity field exhibits the same broadening in the southern half-basin and the double boundary-layer structure in the northern half-basin. The point of maximum vorticity is north of mid-latitude.

Inertial processes in the southern region serve to broaden the boundary layer and to smooth the flow pattern in the region of incoming flow. Concomitant with this broadening is the decreased contribution of the wind-stress to the balance of the various forces in the outer regions of the boundary layer. Hence, the simple Sverdrup balance, viz., a southward flow in response to a negative wind-stress curl, is valid over a somewhat smaller region.

In the northern half of the basin the linear system requires that, as the vorticity acquired in the interior is dissipated by viscosity in the boundary layer, the particle of fluid moves smoothly out to its equilibrium position, i.e., the position where the particle satisfies the Sverdrup balance as it moves out of the boundary layer. In the presence of nonlinear processes the particle overshoots its equilibrium position because of its inertia and consequently loses more negative vorticity than is required before it moves into the interior. Thus, there is a region of positive vorticity as the particle turns southward to regain the equilibrium position, and here viscous effects cause a dissipation of positive vorticity or, in effect, create negative vorticity. Hence, in this region viscosity serves the same function as the wind-stress curl does in the interior. There is also a smaller overshoot southward before the particle attains its equilibrium position.

We sum up the picture as follows: non-linear processes cause northward advection of negative vorticity everywhere in the southern boundary layer and in a band adjacent to the boundary in the north. In the outer part of the northern half of the boundary layer, nonlinear processes cause a spatial oscillation and in part of the region positive vorticity is advected southward. The net effect of inertial processes is, therefore, to advect negative vorticity northward and thereby to counteract the effect of the Sverdrup type of balance in the interior. We shall see later that as a result of this inertial effect the non-linear flow field involves a transport through the western boundary layer which is less than that which would result from the linear system.

Finally, we notice that the non-linear distortion in the nearly-geostrophic pressure field of Fig. 5 tends to crowd the streamlines (hence the isobars) towards the north. As this effect is increased there will be a stronger geostrophically balanced flow toward the east as a boundary layer is created along the northern boundary.
Fig. 5. Contour lines of $\psi$, $p$ and $\zeta$ are shown for the case $\epsilon = 0.05$, $\sqrt{R} = 0.03$. $\psi_{\text{max}} = 1.836$ and $|\zeta_{\text{max}}| = 229.9$. See legend of Fig 4.

Fig. 6. Contour lines of $\psi$, $p$ and $\zeta$ for the case $\epsilon = 0.05$, $\sqrt{R} = 0.05$. $\psi_{\text{max}} = 1.829$ and $|\zeta_{\text{max}}| = 230.6$. See legend of Fig 4.
The latter feature is evident in Fig. 6 where we show the streamline, pressure and vorticity patterns for the case \( \epsilon = 0.05, \sqrt{R} = 0.05 \), i.e., when the inertial and frictional boundary layer thicknesses are equal. Non-linear effects have become so intense in the western boundary layer that particles are advected all the way up to and along the northern boundary. It is still necessary that the negative vorticity acquired in the interior be lost through viscous dissipation before the particles enter the region of linear, Sverdrup-like balance. Inertial effects are so intense that the particles overshoot their equilibrium positions to a much larger degree than in the previous case and the domain of positive vorticity is much extended. The region of higher pressure near the northwestern corner serves to keep the streamlines close to the boundary while simultaneously allowing for a strong geostrophically balanced flow northward near the western boundary and eastward near the northern boundary. The large region of positive vorticity in the northwest reflects the violent overshoot of the equilibrium position by the particles and the subsequent return to the interior region. The present case differs from the previous one only in the degree of nonlinear effects. Qualitatively the results are the same. Note that the broadening of the southern boundary layer is more in evidence in the vorticity pattern for this case.

Figure 7 is again qualitatively similar to the previous one but it exhibits to a much greater degree the effects of inertia. The inertial boundary-layer thickness* is twice that of the frictional one, i.e., \( \epsilon = 0.05, \sqrt{R} = 0.1 \). The inertially generated current near the northern boundary is as intense as the western boundary current, and practically all of the streamlines go into the northern boundary layer. The region of positive vorticity covers almost the entire upper third of the basin and the north-south asymmetry in the thickness of the region of negative vorticity is more pronounced. The isobar pattern is such that much more of the flow in the outer regions of the “boundary layer” is down the pressure gradient.

(c) Strongly non-linear flows

The flows discussed up to this point appear to be based on the same type of physical processes and the same type of balance among the various forces. When we increase the Rossby number, \( R \), so that the inertial boundary-layer thickness is even larger relative to the frictional boundary layer, the pattern of flow changes drastically. This type of flow is shown in Fig. 8 where \( \epsilon \approx 0.025, \sqrt{R} = 0.1 \), i.e., the inertial boundary-layer thickness is four times the frictional one. (We change to the case where \( \epsilon \approx 0.025 \) in order to keep the thicker of the two boundary layers down to one-tenth of the scale of the basin).

The principal qualitative changes in the pattern are observed to be: (a) the flow in the northern half of the basin is everywhere to the west except for the intense northern boundary current which flows eastward; (b) a boundary layer is generated on the eastern boundary as well as the western and northern boundaries and it extends a considerable distance southward; (c) the flow is very geostrophic and the slightly ageostrophic component of flow is primarily down the pressure

*The reader should keep in mind that the term “inertial boundary-layer thickness” is based on the simple argument of balance of terms. By this stage of the discussion the term has lost much of its physical significance and should be looked upon simply as a relative measure of the degree of nonlinearity.
Fig. 7. Contour lines of $\psi$, $p$ and $\zeta$ for the case $\epsilon = 0.05$, $\sqrt{R} = 0.1$. $\psi_{\text{max}} = 1.797$ and $|\zeta_{\text{max}}| = 166.4$. See legend of Fig. 4.

Fig. 8. Contour lines of $\psi$, $p$ and $\zeta$ for the case $\epsilon = 0.025$, $\sqrt{R} = 0.1$. $\psi_{\text{max}} = 3.164$ and $|\zeta_{\text{max}}| = 216.4$. See legend of Fig. 4.
gradient (reflecting the intensity of the inertial processes); and (d) the region of positive vorticity is now located primarily in the southeastern corner of the basin.

The qualitative differences between the present case and the previous ones can be traced to the role that the advective processes play in the dynamical balance. The reader will recall that in all of the previous cases inertial effects served to transport negative vorticity northward both in the intense boundary-current regions and in the region of positive vorticity. This behavior was opposed to that of the interior region where the negative wind-stress curl was balanced by a uniform southward flow. In this sense, then, one can say that the role of non-linear processes was to modify and counteract the dominant processes in the linear system.

In the present, strongly non-linear, flow negative vorticity is advected northward along the western boundary and southward along the eastern boundary. In the latter region, therefore, the effect of inertial processes is in the same direction as that of the wind-stress. Furthermore in the previous cases inertia caused an overshoot of the particles to positions north of their equilibrium position with respect to the interior region. In this northern location the particles lost an excess amount of negative vorticity. In order to return to the location in the interior where the Sverdrup balance obtains a particle was forced to go into a region of positive vorticity where dissipation caused it to lose the excess positive vorticity and to arrive at the appropriate latitude where its relative vorticity was negligible. In the strongly non-linear case where inertia causes the particle to come southward along the eastern boundary the dynamical picture is completely different. According to the previous argument, if the particle were now to overshoot its equilibrium position, it would come out of the boundary layer with too much positive vorticity and at a latitude south of its equilibrium position. The argument is inconsistent because the vorticity argument requires the particle to go even further to the south and the argument based on equilibrium position requires the particle to go to the north.

The inconsistency is associated with the assumption that the Sverdrup type of flow still obtains in the interior and that a particle must leave the boundary layer and eventually enter a region of Sverdrup-like balance. In fact, in the present case, inertial effects are significant in a large part of the northern half of the basin and many of the particles do not go through a Sverdrup regime. The fact that the amplitude of the stream function is significantly larger emphasizes the argument that the Sverdrup balance does not determine the flow. The point is that the present case is essentially non-linear, and arguments pivoted about the linear case no longer apply. This conclusion is not too surprising—in fact, it is much more surprising that we could pivot the discussion about the linear case for as wide a range of values of $R$ as we have.

The present case looks more like the flow derived by Fofonoff (1954) for free westward streaming in the interior with a northern, inertially controlled boundary layer. He derived his results for a frictionless system in the absence of a wind-stress and the pattern was symmetric about mid-longitude. It is clear that the wind-stress has a considerable effect in our solution, and friction is also important. A flow which looks qualitatively like the present one can be derived by superimposing a weak Sverdrup circulation on the free solution of Fofonoff.

More striking examples of strongly non-linear flow are shown in Fig. 9 for the
cases $\epsilon = 0.018$, $\sqrt{R} = 0.11$, and $\epsilon = 0.025$, $\sqrt{R} = 0.2$, i.e., the inertial boundary layer is respectively 6 and 8 times the frictional one. Here the patterns are more nearly symmetric about mid-longitude and bear some resemblance to Fofonoff’s free solution, although in our cases there are no boundary layers to the east or west. The flows are still remarkably geostrophic.

Fig. 9. Strongly non-linear flows shown as contours of $\psi$ for the cases $\epsilon = 0.018$, $\sqrt{R} = 0.11$ (upper) and $\epsilon = 0.025$, $\sqrt{R} = 0.2$ (lower). For the upper case $\psi_{\text{max}} = 7.132$ and for the lower case $\psi_{\text{max}} = 12.81$. 
(d) Magnitude of mass transport

It was mentioned earlier that the mass transport of our model ocean char
with the value of the Rossby number, \( R \). In Fig. 10 we show the value of \( \psi \)
which measures the total transport of the ocean, as a function of \( \sqrt{R} \) for the \( \epsilon = 0.05 \). For these results the reader should recall that \( \psi \) is non-dimensional by means of

\[
\psi = \frac{\beta d}{W} \psi_{\text{dimensional}}
\]

For the linear system doubling \( \frac{W}{d} \) would double the actual stream function that the non-dimensional \( \psi \) remains constant.

![Fig. 10. The transport through the western boundary layer, \( \psi_{\text{max}} \), is plotted as a function of \( \sqrt{R} \) for the case \( \epsilon = 0.05 \). \( \psi_{\text{max}} \) decreases slowly with increasing \( R \) until \( \sqrt{R} = 0.12 \) after which \( \psi_{\text{max}} \) shows a rapid increase. For \( \sqrt{R} > 0.12 \) the flow is strongly non-linear.](image)

The value of \( \psi_{\text{max}} \) at \( \sqrt{R} = 0 \) corresponds to the Sverdrup transport. Note that as \( \sqrt{R} \) increases, the value of \( \psi_{\text{max}} \) decreases (hence the transport is less than the Sverdrup transport) until \( \sqrt{R} \) achieves a value corresponding to strong non-linear flow. For larger values of \( \sqrt{R} \), \( \psi_{\text{max}} \) increases sharply. This result reflects the effect of nonlinear processes. As long as the flow is not of the strongly non-linear type, the effect of inertia is to carry negative vorticity northward and thereby offset the effect of the wind. When the boundary layer reaches the eastern boundary, inertia acts in the same sense as the wind, i.e., negative vorticity is advected southward, and the transport begins to increase sharply.

4. BALANCE OF TERMS IN THE VORTICITY EQUATION

Much of the value of a numerical investigation of the type presented in this paper is that one can study the balance of terms in the equations at each point in the system. Such a study is especially instructive in our problem because of the relatively small number of parameters of the basic model.
The balance of forces in the linear case is easily understood, especially if one analyzes the problem with boundary-layer methods. As we have seen in Part 1, the interior region involves a Sverdrup type of flow, i.e., a balance between the $\beta$-term and the curl of the wind-stress. In the western boundary current the $\beta$-term is balanced by frictional forces. There is, of course, a transition region where the boundary current merges with the interior.

It is instructive to look at the balance of forces in the non-linear flows as well. An auxiliary program was written to print out the percentage contribution of each term in the vorticity equation at every grid point. Let

$$\Sigma = R|\mathbf{v} \cdot \nabla \zeta| + |v| + \epsilon |\zeta| + |\text{curl} \tau|$$

and then denote the percentage contribution of each term in the vorticity equation to the total balance as follows:

$$N = \frac{R \mathbf{v} \cdot \nabla \zeta}{\Sigma}, \quad \beta = \frac{v}{\Sigma}, \quad F = \frac{\epsilon \zeta}{\Sigma}, \quad W = -\frac{\text{curl} \tau}{\Sigma}. \quad (4.1)$$

Thus $N$, $B$, $F$, and $W$ respectively correspond to the percentage contribution of inertial forces, $\beta$-term, frictional dissipation and wind-stress curl to the balance in the equation

$$N + \beta + F + W = 0 \quad (4.2)$$

For the linear problem we write the balances as

interior : $\beta \sim W$ \hspace{2cm} (4.3)

boundary layer : $\beta \sim F$

The non-linear system is much more difficult to discuss because the boundaries of the different regions must be delimited and, in general, the balance of forces involves more than two of the four terms. Therefore, we proceed with the discussion by first considering the slightly non-linear case in detail.

(a) Case : $\epsilon = 0.05, \quad \sqrt{R} = 0.03$

For this case it is useful to keep in mind the results from the perturbation treatment of the inertial terms in Part 1. The reader will recall the effect of inertia in the southern and northern halves of the western boundary layer. In the south, vorticity is advected into regions of more negative vorticity hence $\mathbf{v} \cdot \nabla \zeta < 0$. Since negative vorticity is being dissipated, $\epsilon \zeta < 0$. Hence, these two terms re-enforce each other and together balance the $\beta$-term and we have

$$\beta \sim F + N \quad \text{(balance of type 1)} \quad (4.4)$$

In the north, vorticity is advected into regions of more positive vorticity so that $\mathbf{v} \cdot \nabla \zeta > 0$. The remaining terms have the same sign, hence,

$$F \sim \beta + N \quad \text{(balance of type 2)} \quad (4.5)$$

Flow of type 2 is strongly dissipative and it is in this region that most of the vorticity accumulated in the interior is dissipated. As the system becomes more and more non-linear, the point of maximum amplitude of vorticity is shifted northward and the north-south extent of the region in which (4.4) is valid increases while that with the balance given by (4.5) shrinks.
Furthermore, we have seen that inertial effects widen the western boundary layer in the southern part of the basin. In the north the region where (4.5) obtains is narrow but inertial effects create a countercurrent region offshore. As the streamlines turn to come southward the meridional velocity is small and the amplitude of the vorticity is also smaller because the vorticity must change from negative to positive values. The net result is that the inertial term is largest and it balances the rest. Thus

$$N \sim F + \beta \quad \text{(balance of type 3)} \quad (4.6)$$

In the countercurrent region the vorticity is positive, the flow is southward, and particles are advected into regions of positive vorticity. Consequently, we have the balance given by (4.4) but the signs of the terms are reversed. We denote this region by

$$\beta \sim F_+ + N \quad \text{(balance of type 4)} \quad (4.7)$$

where $F_+$ indicates that $\xi > 0$.

As we noted in section 3, inertial effects also cause the southward-going particles to overshoot their equilibrium positions and there will consequently be another region where the flow is slightly northward as the particles come into their equilibrium positions. This region is very narrow and seems to be one of detail. Also, the percentage contribution of the wind-stress in this region is comparable to the other contributions and it seems best to denote this region as a transition to the interior region. We need only note that, as a particle comes out of the boundary layer, there will be a region of spatial oscillation which reflects the effects of inertia. The amplitude of the oscillation is small. For cases with less friction the oscillation could have a larger amplitude.

The interior region, of course, still contains Sverdrup-type flow ($S$).

The above discussion is summarized in Fig. 11a where we sketch the boundaries of the different regions. The region of transition to Sverdrup flow is shaded.

(b) Case : $\epsilon = 0.05$, $\sqrt{R} = 0.05$

When the inertial and frictional boundary-layer thicknesses are equal, a current forms at the northern boundary. The balances of forces are qualitatively similar to those of the previous case but the extent of the regions is altered; e.g., the point of maximum vorticity, hence the transition boundary between flows of types 1 and 2, is also farther north. There is a broader region with flow of type 3 near the northern boundary. On the offshore side of the region of type 1 flow there is a band of (strongly dissipative) type 2 flow through which all streamlines pass. Figure 11b exhibits the results for this case.

(c) Case : $\epsilon = 0.05$, $\sqrt{R} = 0.1$

Here the inertial effects are so strong that flow of type 3 exists along almost the entire length of the northern boundary. In fact, the balance at the grid-points next to the northern boundary is almost entirely between inertial and frictional terms. As we move southward from the northern boundary, flow of type 3 is modified by an increasingly important contribution by the curl of the wind-stress. As the latter becomes stronger the contribution of nonlinearity becomes weaker.
We denote this transition region by $3W$. Also, for this case the point of maximum amplitude of vorticity is at the northwest corner so that almost the entire western boundary layer has a flow of type 1 with only a small region near the northwest corner of type 2.

In Fig. 11c we have included a second shaded region, $4W$, in which flow of type 4 is modified by a sizable contribution from the wind-stress. Here the ratios $N/B$ and $F/W$ reach values as high as 0.5. The principal balance is still of the Sverdrup type but this simple flow is modified significantly. In fact, $F/W > 0.25$ over almost the entire northern quarter of the basin. The shaded transition region to Sverdrup flow is offshore of regions 1 and 2.

(d) **Case:** $\epsilon = 0.018, \sqrt{R} = 0.11$

In Fig. 11d we analyze the strongly non-linear system. The different types of regions have already been discussed and we note simply that there is no longer any obvious boundary-layer structure left in the system. There are still isolated regions of flows of types 1, 2, and 3, but they are broad and the shaded transition region to the Sverdrup solution is very broad. In fact, the Sverdrup flow itself is modified by nonlinearity and friction. Frictional effects are distributed over much of the basin and the region, 2, of very strong dissipation is actually quite small.
(e) Problems for constructing an analytical model

The foregoing results give some indication of the difficulties with which one is confronted in trying to construct an analytical model. In Fig. 11a we note that the western boundary layer consists of flows of type 1 in the south and 2 in the north. In both regions the principal balance is between friction and the $\beta$-effect with inertia processes enhancing first one effect and then the other. As we move away from the western boundary in the northern part of the basin, there is a region (3) where nonlinear terms are dominant. Then we enter a region (4) where the balance is again like that of region 1 but where all of the terms have the opposite sign. To fit all four of these regions into a boundary-layer analysis and satisfy the necessary matching conditions is not a simple task. For this particular case, however, a perturbation treatment of the nonlinear terms is possible.

Figure 11b contains broader regions of flows of different types and perturbation analysis no longer works. Hence, one must solve the problem by some other approximate method. The boundary-layer technique offers a slight simplification of the general problem but too little to make the problem tractable. The cases shown in Figs. 11c and d appear to be even more formidable.

Surprisingly enough it is possible that the strongly non-linear case, even more non-linear than that shown in Fig. 11d, may be amenable to a partially analytic technique. The system seems to be evolving toward a regime where the basic solution is similar to Fofonoff’s free solution but with modifications due to friction and wind-stress. Because the amplitude of the stream function is large, one must take the amplitude into account in an approximate balance. It appears that a possible approach may be to take as a zero-order problem (subscript zero)

$$ v_0 \cdot \nabla (R\xi_0 + f) = 0 $$

or

$$ R \nabla^2 \psi_0 + f = F(\psi_0) \quad (4.8) $$

where $F(\psi_0)$ is an arbitrary function of the stream function and then to take

$$ v_0 \cdot \nabla (R\xi_1 + f) + v_1 \cdot \nabla (R\xi_0 + f) = -\epsilon \nabla^2 \xi_0 + \text{curl} \tau \quad (4.9) $$

as the next order problem (subscript 1) to determine the indeterminate part of $\psi$ which arises from $F(\psi_0)$ in (4.8).

The two parts of the problem turn out to be linear and thus may be tractable. It may not be necessary to solve equation (4.9). Possibly just finding the solubility conditions for the equation may suffice to resolve the indeterminacy involved in $F(\psi_0)$ and thereby yield a minimal result.

5. FLOW IN REGIONS WITH ZERO CURL OF THE WIND-STRESS

It was shown in Part 1 that the Sverdrup interior implies that, if the wind-stress curl vanishes at some latitude, the associated southward velocity also vanishes there and the latitude can be considered to behave effectively like a boundary. Nonlinear effects alter this simple picture.

In order to investigate the effects of inertial processes on this apparent boundary for the linear flow, the following case was treated. The wind-stress curl was chosen to be the sickle function shown in Fig. 12d, i.e.,
Fig. 12. Contours of the stream function for $\varepsilon = 0.05$ are shown for cases: (a) $\sqrt{R} = 0.001$; (b) $\sqrt{R} = 0.05$; (c) $\sqrt{R} = 0.14$. The curl of the wind-stress is purely zonal and has the shape of the sickle function shown in Fig. 12d.
\[
curl \tau = -\frac{8}{3\pi} \sin y, \quad 0 \leq y \leq \frac{3\pi}{4}
\]
\[= 0, \quad \frac{3\pi}{4} < y < \pi
\]
where \(8/3\pi\) is chosen so that \(\int_0^\pi \int_0^y \text{curl} \tau \, dx \, dy\) has the same value as that in the previous section.

The final, steady, flow patterns with \(\epsilon = 0.05\) are shown in Figs. 12a–c for the cases \(\sqrt{R} = 0.001\), \(\sqrt{R} = 0.05\), and \(\sqrt{R} = 0.14\). For the first (nearly linear) case we see that the streamline pattern is nearly completely confined to the lower three-fourths of the basin. There is a slight extension into the northern quarter-basin because of the discontinuity of \(\text{curl} \tau\) at \(y = \frac{3\pi}{4}\). However, for practical purposes the flow is confined to the region \(y \leq \frac{3\pi}{4}\).

When \(\sqrt{R} = 0.05\), i.e., the inertial and frictional boundary layers have the same thickness, the streamlines pattern is that which is shown in Fig. 12b. We note that the western jet extends up into the region of zero curl and that the behaviour of the system is qualitatively similar to that which was discussed in the previous section. There is still a broad region of quiescent water in the northeast. The beginning of a spatially oscillatory behaviour is evident in the streamline pattern around latitude \(y = \frac{3\pi}{4}\). The oscillation appears to be associated with the fact that there is a broad region of water which is not directly forced and which can readily yield to the dynamical pattern of flow.

We show in Fig. 12c the case where \(\sqrt{R} = 0.14\). The jet extends to the northern boundary and the flow again has a pattern similar to that of the previous section. The absence of a wind in the northern quarter of the basin serves only to delay the onset of a strongly nonlinear flow. Certainly the jet seems to be uninhibited in extending to the northern region. We conclude from these results that for the barotropic ocean model some other feature, such as a region of wind-stress curl of opposite sign to the north or perhaps bottom topographical features, must be present in the system to inhibit the jet from extending into the north.

6. SOME REMARKS

(a) Transient vs. steady oceanic response

In an earlier paper (Veronis, 1966) an approximate solution to the present problem was derived by means of a truncated system of Fourier components. One of the principal results of that paper was that the ocean could have a transient response to a steady wind-stress. In view of the present results the former result appears to be due to the severe truncation of the Fourier representation.

Bryan (1963) also derived a transient oceanic response to a steady wind when the system became sufficiently nonlinear. His numerical study involved an even finer finite-difference grid than the one used in the present study, so there is no question in his case of transient behaviour due to truncation of the system. However, Bryan used the Navier–Stokes form of friction and satisfied a boundary condition of zero velocity along the boundary. This condition yields a profile with a strong inflection point and it is very likely that the transient response of his model is due to shear flow instability of the flow near the boundary. That, in fact, is the cause attributed by Bryan to the transient response.
The statistically steady response of the ocean to a fluctuating wind is still to be studied by means of a finite-difference analysis. Such an investigation has been undertaken and will be reported in a future paper.

(b) *Separation of the western boundary current from the coast*

The western boundary current separates from the coast in both the Atlantic and Pacific Oceans. The irregularities of the boundaries may play some role in this process. Bryan (1963) has investigated a specific effect of irregular boundaries by treating a case with the basin wider in the southern half-basin and an abrupt discontinuity to a narrower basin in the northern half. No break of the boundary current from the coast occurred in his investigation.

It seems clear that the barotropic model is unlikely to yield separation. The pressure field associated with the flow pattern requires a high pressure region adjacent to the boundary to maintain the geostrophically balanced flow along the boundary. Separation would require either a completely different pressure distribution or a dynamical balance which allowed the flow to go through the high pressure region. Neither possibility has been indicated by the present study and it is difficult to see how a barotropic model could yield such a result.

It seems much more likely that separation is associated with baroclinic effects. Certainly both lateral and bottom irregularities could give rise to initial separation but the integrity of the current after it leaves the region where tropographical effects are important points strongly to baroclinic effects.

(c) *The significance of the results from a barotropic model*

It was stated earlier in this paper that the barotropic model is an idealized model of the ocean. It is not an approximation in the sense that it can be derived as a zero-order or first-order system in an expansion or a series of expansions in terms of known small parameters. Some of the complicating physical processes must be simply assumed out of the system. Since this procedure does not lead to a sequence of models which converges to the "real" ocean, we can legitimately ask: What value does such a model have? How are we to interpret the results of the model? Are there measurable properties of the ocean to which derived quantities from the model correspond?

The difficulties in giving explicit answers to the above questions are evident from simple considerations; e.g., in treating friction as a bottom drag law we make the implicit assumption that the velocity at the top of the Ekman layer is related to the vertically averaged velocity in some simple way. Observational evidence indicates that reasonably intense bottom currents flow in the direction of the surface currents in some areas and in the opposite direction in others. Thus in some regions one errs in the sign of the transfer process when one uses a bottom drag law. The same error (one of sign) occurs when one treats the frictional process in terms of a Navier–Stokes law with an Austausch coefficient. Webster (1961) has analyzed some velocity data from the vicinity of the Gulf Stream and has shown that for that set of data the transfer of momentum takes place from the fluctuations to the mean field so that the Austausch coefficient should be negative.

The main point is, however, that some dissipation must be present in the barotropic model and, even if it is represented through an over-simplified process,
the principal physical effect is there. So even if we err in detail we have the necessary physical process incorporated in the simple model.

To return to the questions asked above, we note from the foregoing remarks that we have some evidence that details of oceanic processes are grossly simplified and distorted by the simple barotropic model. We have seen in section 4 that it is not possible to confine the circulation to a region where the wind-stress curl has one sign; it is necessary that we add some other feature to the barotropic model to confine the circulation—either bottom topography or a region where the wind-stress has a curl of the opposite sign. It is also likely that baroclinic effects are necessary to provide the integrity of the stream after it leaves the coast.

The decreased role of friction in the formation region of the Gulf Stream and the enhanced effect of frictional processes in the northern region are probably realistic aspects of the barotropic model. The observed countercurrent under the Gulf Stream occurs south of Cape Hatteras. Deep currents in the same direction as the Gulf Stream occur after the current leaves the coast. Qualitatively these observations agree with our conclusions about the relative role of bottom friction in the two regions.

We noted also that as R is increased the flow changes drastically from moderately nonlinear flow to a pattern of flow which we have termed strongly nonlinear. Now it is unlikely that such drastic changes would occur in the real ocean or in a more realistic model. However, what does appear as a distinct possibility is that the recirculation which is characteristic of strongly nonlinear flow may be present in the real ocean. If it is, it is more likely confined to the vicinity of the Gulf Stream and, if we take the barotropic model as guide, it must occur after the Gulf Stream leaves the coast. Exactly how the real ocean confines the recirculation to this small region (if it does) cannot be determined from a barotropic model alone.

The one type of response which carries over most directly from the barotropic model to the real ocean is probably the transient behaviour. We have seen that nonlinear effects require that we extend our study to entire oceanic basins rather than to treat basins delimited by the sign of the wind-stress. Transient flows also require a study over a more extensive region. We shall return to this study in a future paper.

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