Conservation Equations in Physical Oceanography
An Introduction

• The goal- in order to describe the state of the ocean at any given place \((x,y,z)\) and time \((t)\) we need to know:
  - the velocity field- \(u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)\)
  - temperature and salinity \(T(x,y,z,t), S(x,y,z,t)\)
  - pressure and density \(p(x,y,z,t), \rho(x,y,z,t)\)

• Therefore, to find 7 unknowns we need 7 equations...

• We already discussed 2 equations:
  - The Equation of State (EOS): \(\rho=\text{func}(T,S,p)\)
  - The Hydrostatic Relation: \(p=\text{func}(\rho,z)\)

\[
p(x, y, z) = \int_0^z \rho(x, y, z) \, gdz
\]

So, we need to develop (at least) 5 more equations
Conservation Equations:

- Eq#1: Conservation of mass → \textit{continuity equation}
- Eq#2: Conservation of heat → \textit{temperature equation}
- Eq#3: Conservation of salt → \textit{salinity equation}
- Eqs#4-6: Conservation of momentum → \textit{equations of motion (3 equations for u,v,w)}
- That’s 6 equations + density equation (EOS) = 7
  (w-Eq. → Hydrostatic Eq. ; see later…)
Some questions that can be answered from those equations:

• We discussed how surface heat flux can change temperature and how salt/water flux can change salinity, but how can we account for advection by ocean currents?
  (need equations that connect $u,v,w$ with $T&S$)

• How does wind generate ocean currents, and how do the currents change with depth?
  (need equations that connect $u,v,w$ with wind & friction)

• What drives large-scale ocean circulation patterns?
  (hint: Earth rotation is important)

• Can we predict how the ocean state changes with time?, and maybe how it will change in the future?
  (e.g., like weather forecasting, climate prediction, etc.)
So how do we answer those equations:

• Method: develop set of mathematical equations that take into account all the physical mechanisms behind oceanic processes (e.g., gravity, earth’s rotation, friction, other forces, etc.)

• So what is the problem?: the result is a set of **partial differential equations** that are quite complex!

• In fact, there are only a few simple cases where we can actually solve the equations… but we’ll deal with solutions later…

• Note: the differential equations describe *changes* in variables, say how temperature changes with time \(\frac{dT}{dt}=\ldots\) not what the exact temperature is at specific place and time.

So, before we start with the equations, let’s first review some basic calculus relations and terminology that will help us in the development of the equations.
Definition of The Derivative

The equations describe oceanic changes - therefore, they involve derivatives.

The derivative of the function \( f(x) \) at the point \( x_0 \) is given and denoted by

\[
\frac{df}{dx}(x_0) = f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]

The derivative is simply the slope at point \( x_0 \) of the function \( f(x) \).
Some complex processes such as ocean mixing will require use of:

**Higher Order Derivatives**

Let $y = f(x)$. We have:

Second Derivative is: \[ \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2} = f''(x) = y'' \]

Third Derivative is: \[ \frac{d}{dx} \left( \frac{d^2 y}{dx^2} \right) = \frac{d^3 y}{dx^3} = f'''(x) = y''' \]

$n^{th}$ Derivative is: \[ \frac{d}{dx} \left( \frac{d^{n-1} y}{dx^{n-1}} \right) = \frac{d^n y}{dx^n} = f^{(n)}(x) = y^{(n)} \]

First derivative describes the **slope**

Second derivative describes the **curvature**

- large slope
- small slope
- positive slope
- negative slope
Partial derivatives are often used in physical oceanography because most properties are function of more than one variable [e.g., $T(x,y,z,t)$]. $\frac{\partial T}{\partial t}$ would be the change in temperature with time at a fixed location; $\frac{\partial T}{\partial y}$ would be the change of temperature with latitude and $\frac{\partial U}{\partial x}$ the acceleration in the east-west direction.
Some useful notations:

- **Velocity vector** \( V = ui + vj + wk \)
  
  \((i, j, k)\) are unit vectors in \(x, y, z\) directions

The operator **del** of a vector is a vector defined by

\[
\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}
\]

\[
\nabla_h = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}
\]

The operator **dot** of two vectors is a scalar defined by

\[
A \cdot B = a_1b_1 + a_2b_2 + a_3b_3
\]

Therefore:

\[
\nabla \cdot V = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (u, v, w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad [= 0 \text{ is Continuity Eq}]
\]

The operator **cross** of two vectors is a vector perpendicular to both vectors

\[
\vec{A} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}
\]

[used in terms involved Earth's rotation]
Other useful notations in physical oceanography:

**Total derivative** is defined by

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla
\]

So a conservation equation of property \( C \) can be written as:

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial C}{\partial t} + (V \cdot \nabla)C = 0 \quad \text{or} \quad \frac{D}{Dt}(C) = 0
\]
Next class:

- Conservation of mass and the Continuity Equations
- Application for upwelling/downwelling systems