# Wave Radiation Stress

What are they?

- Proposition I: The vertically integrated stress terms derived by Longuet-Higgins and Stewart (1962, 1964; hereafter L-HS) and Phillips (1966, hereafter OP) are correct to order $(\boldsymbol{ka})^2$. They are *relatively* simple to derive and understand.

- Proposition II: Vertically dependent stress terms should, as a necessary condition, conform to L-HS and OP after vertical integration. Evidently, they are not so simple to derive.

# Steady wavy wall flow

For steady irrotational flow in two dimensions, one has

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad \text{and} \quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0 \qquad \text{(1a, b)}$$

and

$$p+\frac{u2+w2}{2}+gz=B$$

where $B$ is the Bernoulli constant. Simplified for deep water, solutions to (1a, b) are

$$u = -u_0 + \tilde{u}; \;\; \tilde{u}(x,z) = kau_0 \mathrm{e}^{kz} \cos kx$$

$$w = \tilde{w}; \;\; \tilde{w}(x,z) = kau_0 \mathrm{e}^{kz} \sin kx$$

where $u_0$ is a constant. The stream function is

$$\psi = -u_0 z + au_0 e^{kz} \cos kx$$

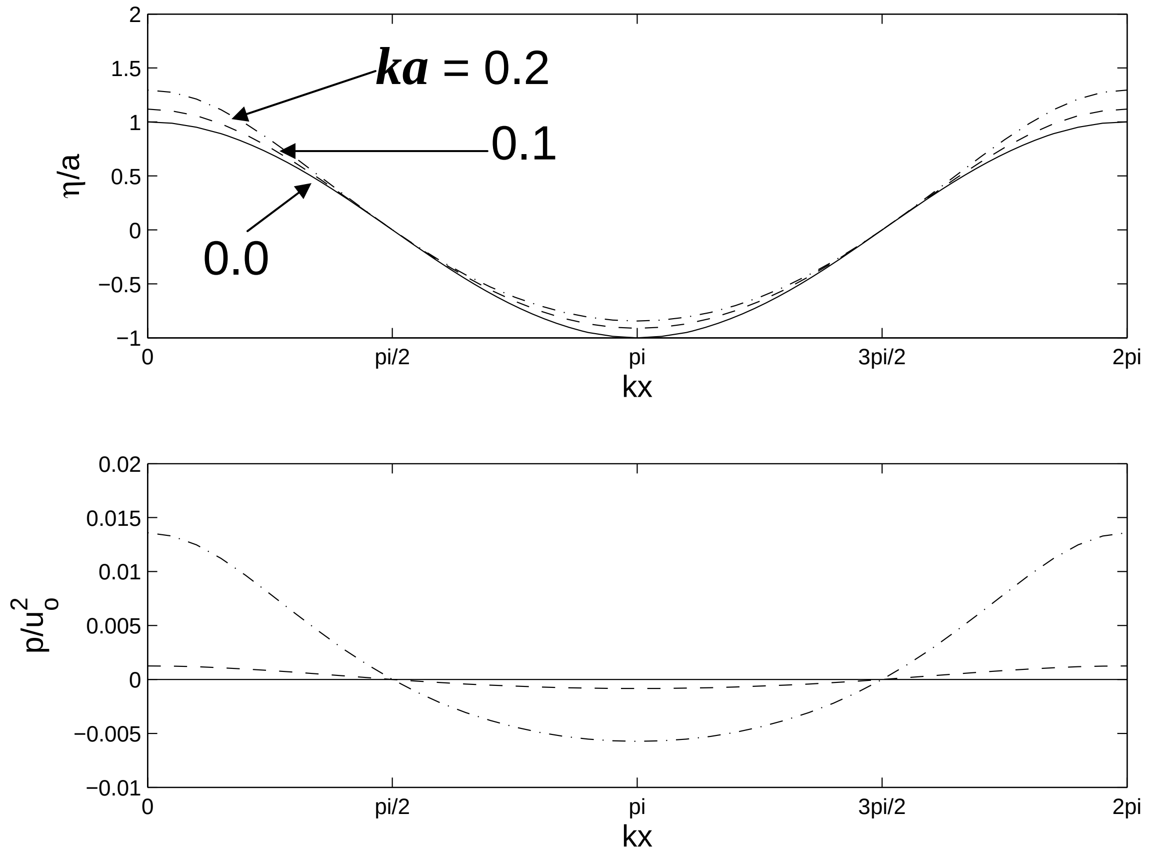
Let $z = \eta(x)$ and $\psi = 0$ so that

$$\eta = ae^{k\eta} \cos kx$$

and from Bernoulli's equation

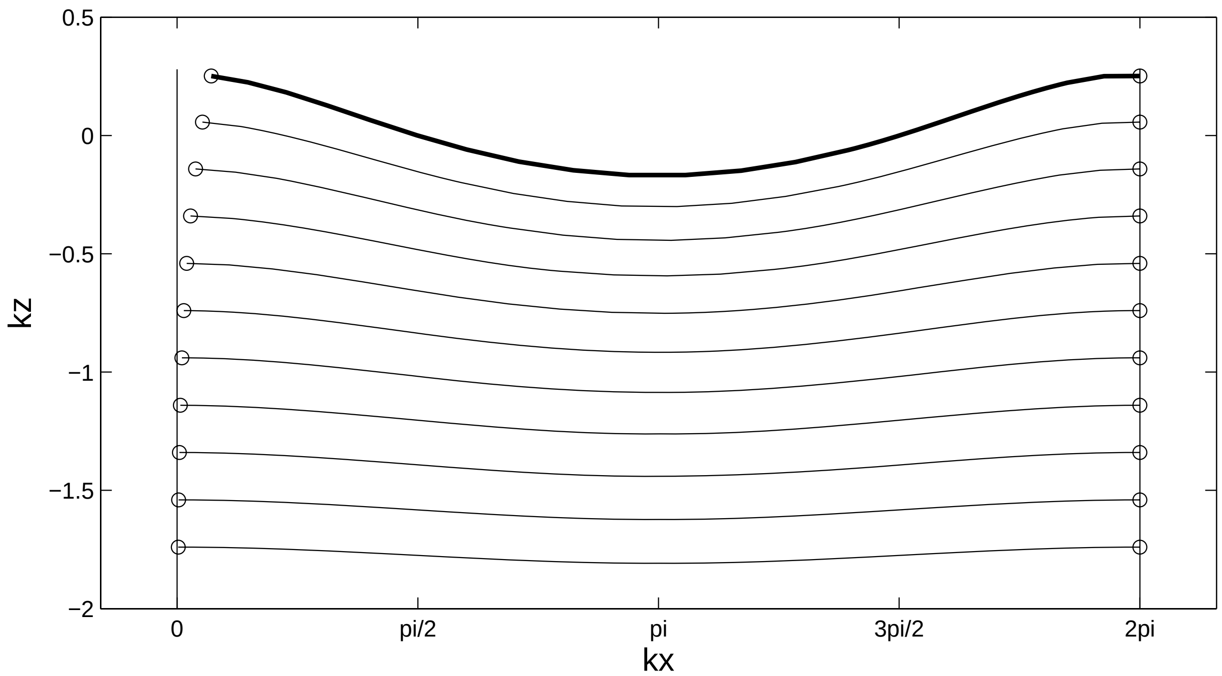
$$p(x,z) = -\frac{u_o^2}{2} + u_o^2 kae^{kz} \cos kx - u_o^2 (ka)^2 \frac{e^{2kz}}{2} - gz + B$$

# Wall shape, $\eta(x)$, and wall pressure, $p(\eta, x)$



# Stokes Drift

$$dx_p / dt = u(x_p, z_p); \quad dz_p / dt = w(x_p, z_p)$$




$$u_0^2 = g / k$$

# Progressive surface wave flow

Dispersion relation: $c^2 = g/k$ so that $c = u_0$

Effect Galilean transformation $x \to x' - ct;\ u \to u' - c$ . Thus,

$$\tilde{u} = kace^{kz}\cos\psi \qquad \tilde{w} = kace^{kz}\sin\psi \qquad \text{(2a,b)}$$

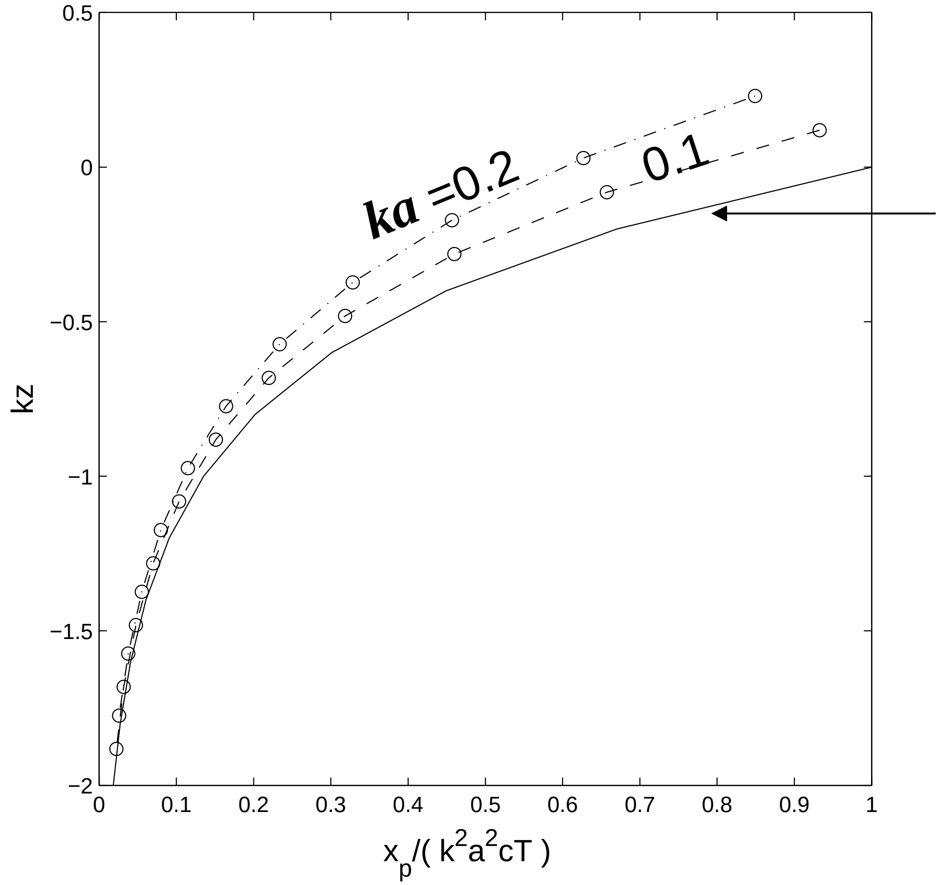
$$\tilde{\eta} = ae^{k\eta}\cos\psi \qquad \text{(3)}$$

where $\psi \equiv kx - \sigma$ and

$$p(x,z,t) = c^2 kae^{kz}\cos\psi - c^2(ka)^2\frac{e^{2kz}}{2} - gz + B \qquad \text{(4)}$$

N.B. Currents = 0

# Stokes Drift




$$x_p/(ka)^2cT = e^{2kz}$$

Separate ***p*** into a wavy part

$$\tilde{p}(x,z,t) = gae^{kz}\cos\psi$$

and a phase-averaged part

$$\overline{p} = -c^2(ka)^2\frac{e^{2kz}}{2} - gz + B$$

What is ***B*** ??

where $\overline{(\ )} \equiv (2\pi)^{-1}\int_0^{2\pi}(\ )d\psi$ . Also, to lowest order in ***ka***

$$\tilde{\eta}(x,t) = a\cos\psi$$

# In pursuit of *B*

The vertical component of the momentum equation is

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}=-\frac{\partial p}{\partial z}-g$$

Integrate vertically from arbitrary ***z*** to $z=\tilde{\eta}$ so that

$$\frac{\partial}{\partial x}\int^{\eta} uw\,dz - uw|_{\eta}\frac{\partial \eta}{\partial x}+\tilde{w}^2(\eta)-\tilde{w}^2(z)+\frac{\partial}{\partial t}\int^{\eta}\tilde{w}\,dz-\tilde{w}(\eta)\frac{\partial \eta}{\partial t}+g(\eta-z)=p(z)-p(\eta)$$

Now $p(\eta)=p_{atm}$ and after phase averaging

$$\bar{p}+\overline{\tilde{w}^2}+gz=p_{atm}$$

$$\boldsymbol{B}=p_{atm}$$

Whereas the instantaneous pressure is continuous across the air-sea interface, the phase-averaged pressure is discontinuous.

# Wave Radiation Stress

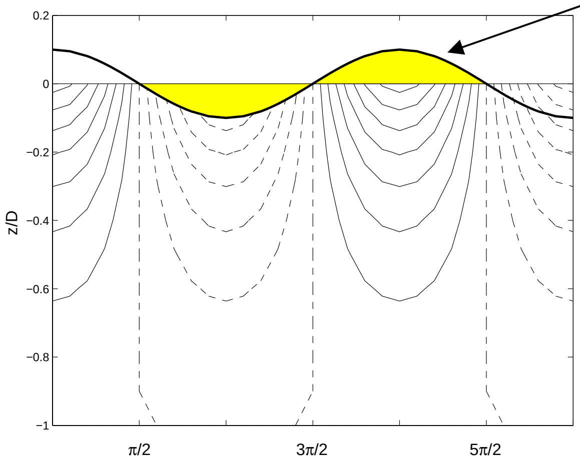
From phase averaged horizontal momentum component, does

$$\frac{\partial}{\partial x}\left(\overline{p}+\overline{\tilde{u}^2}\right)=0$$

where $\overline{p}+\overline{\tilde{w}^2}+gz=p_{atm}$ ? Not quite !

There is a contribution from the elevation: i.e.,

$$\overline{\int_0^{\tilde{\eta}} p dz}=\overline{\int_0^{\tilde{\eta}} gz dz}=g\overline{\tilde{\eta}^2}/2=E/2$$



# Therefore

$$\frac{\partial}{\partial x} S_{xx} = 0; \;\; S_{xx} \equiv \overline{\tilde{u}^2} - \overline{\tilde{w}^2} + \frac{g\overline{\tilde{\eta}^2}}{2}\delta(z)$$

or in 2 horizontal dimensions

$$\frac{\partial}{\partial x_\beta} S_{\alpha\beta} = 0; \;\; S_{\alpha\beta} = \frac{k_\alpha k_\beta}{k^2}\overline{\tilde{u}^2} - \delta_{\alpha\beta}\left\{\overline{\tilde{w}^2} - \frac{g\overline{\tilde{\eta}^2}}{2}\delta(z)\right\}$$

# Voila`