Wave Radiation Stress

What are they?
• **Proposition I**: The vertically integrated stress terms derived by Longuet-Higgins and Stewart (1962, 1964; hereafter L-HS) and Phillips (1966, hereafter OP) are correct to order \((ka)^2\). They are relatively simple to derive and understand.

• **Proposition II**: Vertically dependent stress terms should, as a necessary condition, conform to L-HS and OP after vertical integration. Evidently, they are not so simple to derive.
Steady wavy wall flow

For steady irrotational flow in two dimensions, one has

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \text{and} \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \]  

(1a, b)

and

\[ p + \frac{u^2 + w^2}{2} + gz = B \]

where \( B \) is the Bernoulli constant. Simplified for deep water, solutions to (1a, b) are
\[ u = -u_0 + \tilde{u}; \quad \tilde{u}(x,z) = kau_0 e^{kz} \cos kx \]

\[ w = \tilde{w}; \quad \tilde{w}(x,z) = kau_0 e^{kz} \sin kx \]

where \( u_0 \) is a constant. The stream function is

\[ \psi = -u_0 z + au_0 e^{kz} \cos kx \]

Let \( z = \eta(x) \) and \( \psi = 0 \) so that

\[ \eta = ae^{k} \eta \cos kx \]

and from Bernoulli’s equation

\[ p(x,z) = -\frac{u_0^2}{2} + u_0^2 kae^{kz} \cos kx - u_0^2 (ka)^2 \frac{e^{2kz}}{2} - gz + B \]
Wall shape, $\eta(x)$, and wall pressure, $p(\eta, x)$
Stokes Drift

\[ \frac{dx_p}{dt} = u(x_p, z_p); \quad \frac{dz_p}{dt} = w(x_p, z_p) \]

\[ u_0^2 = \frac{g}{k} \]
Progressive surface wave flow

Dispersion relation: \( c^2 = g/k \)  so that \( c = u_0 \)

Effect Galilean transformation \( x \rightarrow x' - ct; \ u \rightarrow u' - c \). Thus,

\[
\tilde{u} = kae^{kz} \cos \psi \\
\tilde{w} = kae^{kz} \sin \psi
\]

(2a,b)

\[
\tilde{\eta} = ae^k \eta \cos \psi
\]

(3)

where \( \psi \equiv kx - \sigma \)  and

\[
p(x,z,t) = c^2 kae^{kz} \cos \psi - c^2 (ka)^2 \frac{e^{2kz}}{2} - gz + B
\]

(4)

N.B. Currents = 0
Stokes Drift

\[ \frac{x_p}{(ka)^2cT} = e^{2kz} \]
Separate $p$ into a wavy part

$$\tilde{p}(x,z,t) = gae^{kz} \cos \psi$$

and a phase-averaged part

$$\bar{p} = -c^2(ka)^2 \frac{e^{2kz}}{2} - gz + B$$

What is $B$??

where

$$\bar{(\cdot)} \equiv (2\pi)^{-1} \int_0^{2\pi} (\cdot) d\psi$$

Also, to lowest order in $ka$

$$\tilde{\eta}(x,t) = a \cos \psi$$
In pursuit of $B$

The vertical component of the momentum equation is

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} = -\frac{\partial p}{\partial z} - g$$

Integrate vertically from arbitrary $z$ to $z = \tilde{\eta}$ so that

$$\frac{\partial}{\partial x} \int_0^\eta uw dz - uw|_0^\eta \frac{\partial \eta}{\partial x} + \tilde{w}^2(\eta) - \tilde{w}^2(z) + \frac{\partial}{\partial t} \int_0^\eta \tilde{w} dz - \tilde{w}(\eta) \frac{\partial \eta}{\partial t} + g(\eta - z) = p(z) - p(\eta)$$

Now $p(\eta) = p_{atm}$ and after phase averaging

$$\bar{p} + \tilde{w}^2 + gz = p_{atm}$$

$B = p_{atm}$

Whereas the instantaneous pressure is continuous across the air-sea interface, the phase-averaged pressure is discontinuous.
Wave Radiation Stress

From phase averaged horizontal momentum component, does

$$\frac{\partial}{\partial x} \left( \bar{p} + \bar{u}^2 \right) = 0$$

where

$$\bar{p} + \bar{w}^2 + gz = p_{atm}$$

? Not quite!

There is a contribution from the elevation: i.e.,

$$\int_0^\eta pdz = \int_0^\eta gzdz = g\tilde{\eta}^2/2 = E/2$$
Therefore

\[ \frac{\partial}{\partial x} S_{xx} = 0; \quad S_{xx} \equiv \tilde{u}^2 - \tilde{w}^2 + \frac{g\eta^2}{2} \delta(z) \]

or in 2 horizontal dimensions

\[ \frac{\partial}{\partial x} S_{\alpha\beta} = 0; \quad S_{\alpha\beta} = \frac{k\alpha k}{k^2} \tilde{u}^2 - \delta_{\alpha\beta} \left[ \tilde{w}^2 - \frac{g\eta^2}{2} \delta(z) \right] \]

Voila\`